Unit VII

Exponential Functions

Suggested Time

12 hours = 12%

In Mathematics 1201, students worked with linear relations expressed in slope-intercept form, general form and slope-point form (RF6). In Mathematics 2201, they solved problems that involved quadratic equations (RF2). Students will now be exposed to exponential equations of the form $y = a(b)^x$, where $b > 1$ or $0 < b < 1$ and $a > 0$. Work with logarithmic functions will be completed in the next unit.

Exponential function: any function of the form

$$y = a \bullet (c)^x \text{ Where } a>0,\ c > 0,\ c \neq 1$$

Case 1: c > 1 (Exponential Increasing Curve or Exponential Growth Curve)

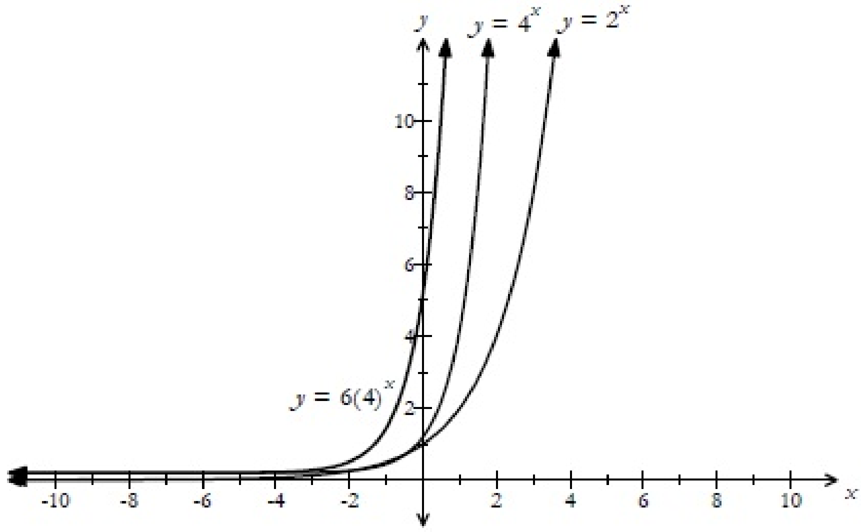
As x increases y increases without bound

ex) Sketch the curves

$$i)\ y = 6(4)^x \quad ii)\ y = 4^x \quad iii)\ y = 2^x$$

for a table of values for x = -3 to 3

x	$y = 6(4)^x$	$y = 4^x$	$y = 2^x$
-3			
-2			
-1			
0			
1	24		
2	96		
3	384		
4			



End Behaviour:
all graphs increase without bound as x gets larger (increasing curves----rises to the right as x increases) these are known as exponential GROWTH curves C>1)
asymptote: the horizontal line an exponential curve does not touch. Here it is the x-axis for all three cases. The equation of the asymptote is y = 0. This means the graph will never hit y = 0.
These curves are all growth curves and the larger the value of C the faster the growth rate. These curves are all exponential growth curves...increasing curves.

Domain: $\{x/x \in R\}$

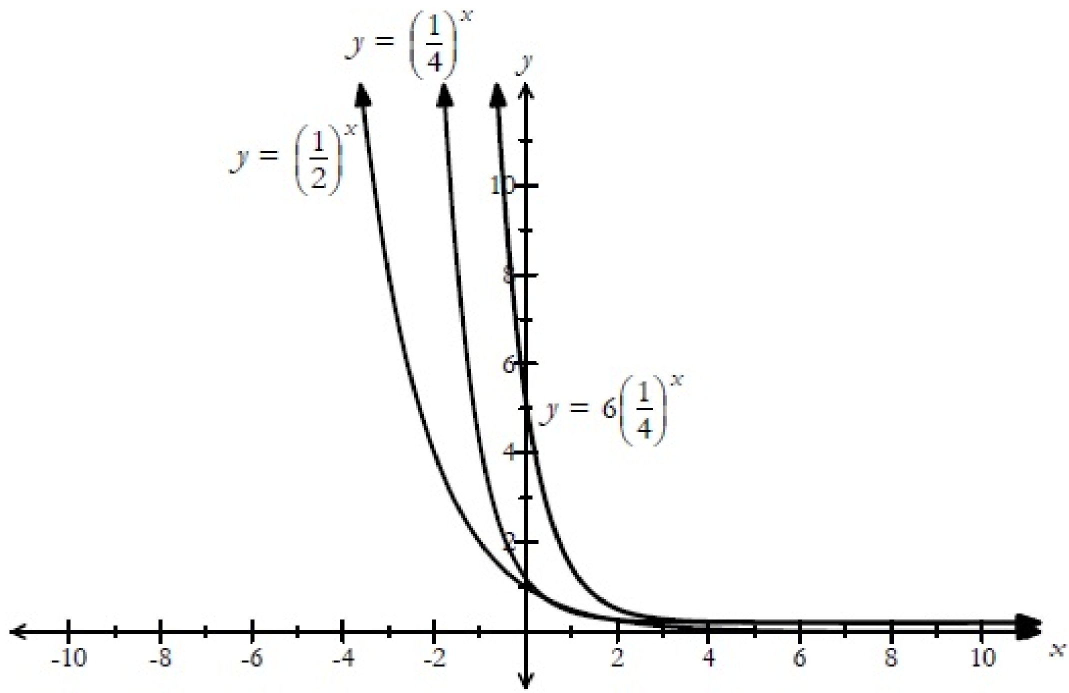
Range: Affected by the Horizontal asymptote [y = 0] y>0

Case II   0 < c < 1 (Decay curves : as x increases y is decreasing forever) $y = a(c)^x$

ex)   Sketch the exponential functions for

$y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{4}\right)^x$, $y = 6\left(\frac{1}{4}\right)^x$

x	$y = (\frac{1}{2})^x$	$y = \left(\frac{1}{4}\right)^x$	$y = 6\left(\frac{1}{4}\right)^x$
-3	8		
-2	4		
-1	2		
0	1		
1	.5		
2	.25		
3	.125		
4	.0625		



End Behaviour: decreasing curves as x increases y is decreasing

Asymptote: y = 0 (Horizontal Line the graphs do not touch or cross)

Domain: {x/x∈R}   Range   {y/y>0, y ∈ R}

Fill in the information for the exponential curves below:

	$y = 2^x$	$y = 3^x$	$y = 5(3)^x$
y-intercept			
number of x-intercepts			
end behaviour			
domain			
range			

	$y = \left(\frac{1}{2}\right)^x$	$y = \left(\frac{1}{4}\right)^x$	$y = 3\left(\frac{1}{4}\right)^x$
y-intercept			
number of x-intercepts			
end behaviour			
domain			
range			

Summary: $y = a \bullet (b)^x$

1) if b > 1 exponential growth (increasing curve)
2) (0,a) = y-intercept
3) if 0 < b < 1, exponential decay curve (decreasing curve)
4) HA: y = 0
5) if b < 0 graph oscillates back and forth between points
6) if b = 1, $y = a \bullet (b)^x$ its just the horizontal line y = a

Transformations of Exponential Functions 7-2

https://www.desmos.com/calculator

$$y = 2^x,\ \ y = 2^x - 4,\ \ y = 2^x + 4$$

1) Vertical Translations: $y = c^x \pm k$

ex) Sketch: On CALC

Summary:
1) all exponential growth curves
2) all translated vertically by "k" units
3) Horizontal Asymptote: y = k
4) Domain: $x \in (-\infty, \infty)$ Range: $y>0$, $y>-4$, $y>4$

Sketch: $y = \left(\frac{1}{2}\right)^x \quad y = \left(\frac{1}{2}\right)^x - 3 \quad y = \left(\frac{1}{2}\right)^x + 2$

1) all exponential decreasing curves
2) same as 2,3,4

2) Horizontal Translations: $y = c^{x \pm h}$

Sketch: $y = 2^{x+2}$ $y = 2^{x-2}$ $y = 2^{x}$

Summary:

1) $y = c^{x+c}$ moves $y = c^{x}$ c units _____

2) $y = c^{x-c}$ moves $y = c^{x}$ c units _____

3) domain: Range: y > 0

4) asymptote: y = 0

3) Vertical Stretch $y = a(c)^{x}$

Stretches:

$y = 3^{x}$ $y = 2 \bullet 3^{x}$ $y = .5 \bullet 3^{x}$

Summary:

1) vertical stretch = |a|
2) domain and range not affected
   (-∞,∞) Y > 0
3) y-intercept = (0,a)
4) asymptote: y = 0

4) Horizontal Stretch: $y = c^{bx}$

$y = 2^x$ $y = 2^{3x}$, $y = 2^{\frac{x}{3}}$, $y = 5^x$ $y = 5^{2x}$ $y = 5^{0.2x}$

$c = 2$

$b = 1$

Summary: 1) Horizontal stretch = 1/b

Reflections with Exponential Functions 7-2 Continued

1) $y = -c^x$

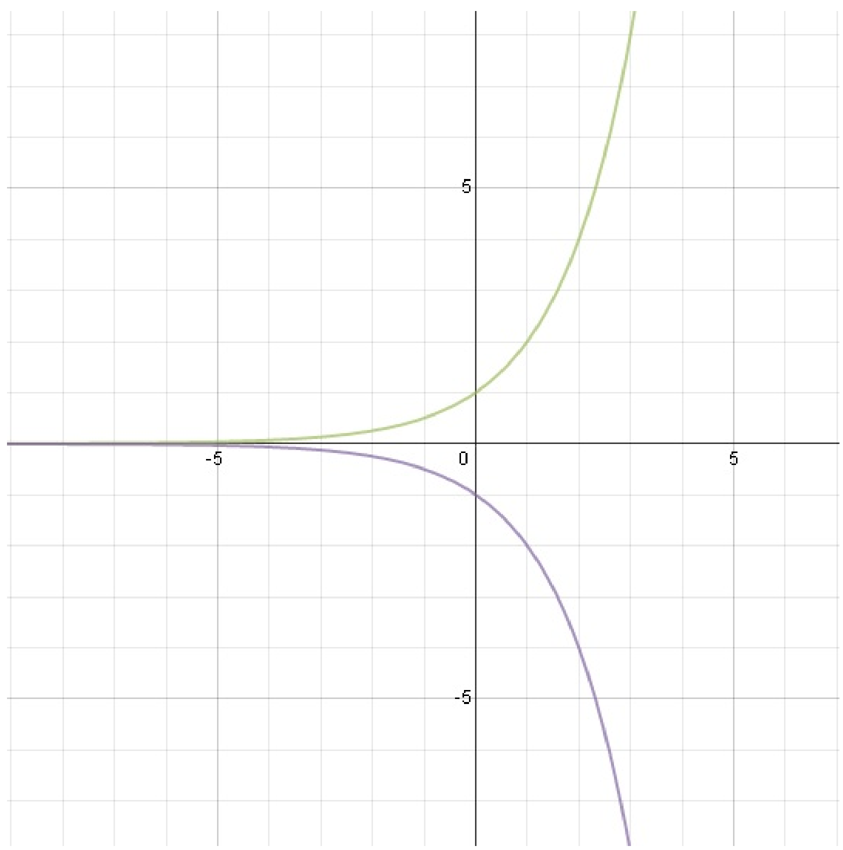
the -ve coefficient will cause the output values (y) to turn negative...hence the reflection will occur in the x-axis whether it is a growth or decay curve

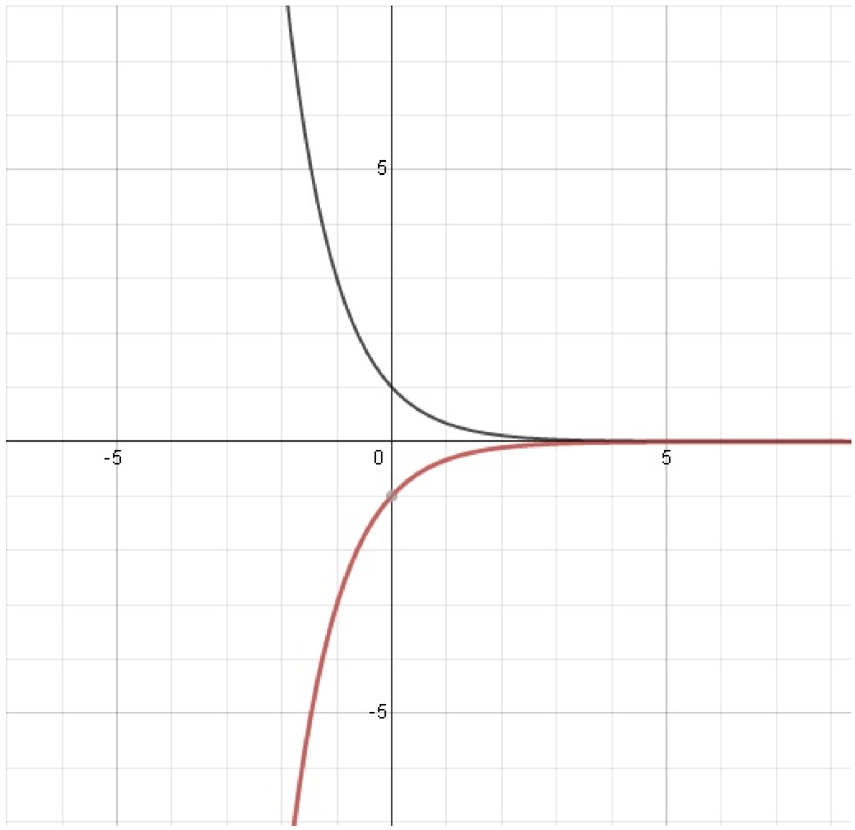
Consider $y = 2^x \quad y = -2^x$, $y = \left(\frac{1}{3}\right)^x$, $y = -\left(\frac{1}{3}\right)^x$, $y = -4^x$

(2,4) → (2,-4) (1,1/3)→(1,-1/3)

Note: we ARE NOT CONSIDERING: $y = (-2)^x$

Summary: Changes Range, y-intercept, and shape in comparison to original





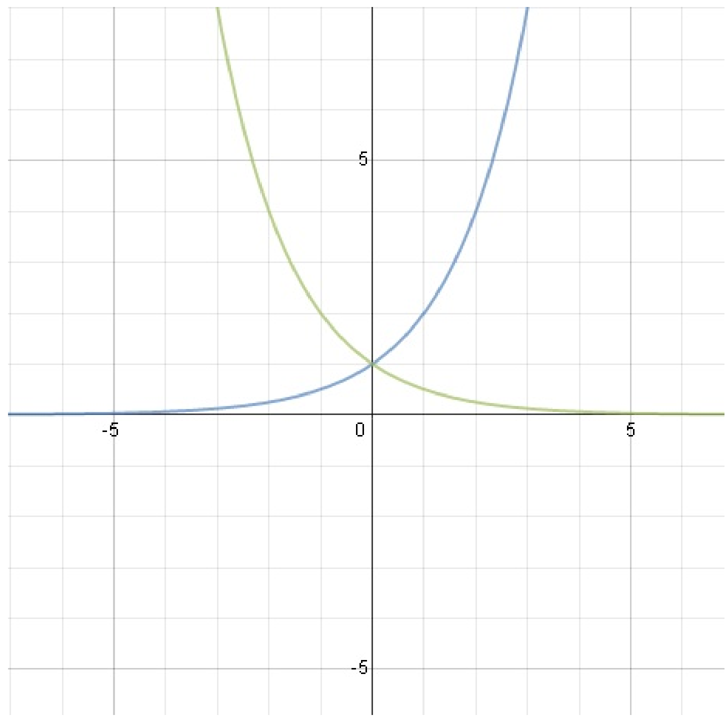
2)

$$y = c^{-x}$$

Consider: $y = 2^x \quad y = 2^{-x} = \left(\frac{1}{2^x}\right) = \left(\frac{1}{2}\right)^x$

causes the input values to switch sign therefore reflection will be in the y -axis (2,4)→(-2,4)

Summary:
1) causes reflection in the y -axis
2) domain x ∈R
3) range y > 0
4) asymptote: y = 0 prerequisite



7-2 Combining effects of all Transformations of Exponential Functions

$$y = a(c)^{b(x-h)} + k$$

1) a = vs
2) c=base (growth or decay)
3) 1/b=HS
4) h= horizontal translation
5) k=vertical translation

Graphing Transformation of Exponential Functions
(Combining All Transformations)

Ex) Determine the domain, range, equation of the horizontal asymptote, where the reflection is, base exponential function, mapping rule for:
(State a,b,c,h,k)

(i) $y=-\frac{1}{3}(2)^{x+3}-1$

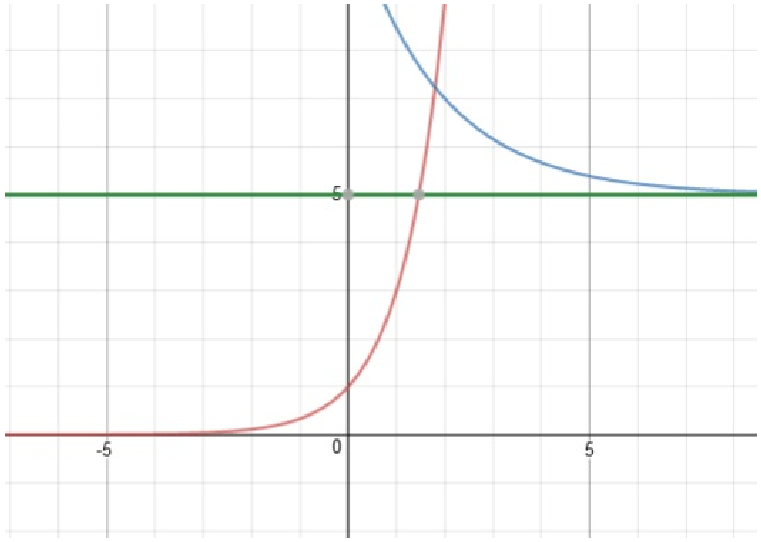
(ii) $y=2(3)^{-\frac{1}{2}(x-2)}+5$

(iii) $y=3(\frac{1}{2})^{\frac{(x-2)}{4}}+8$

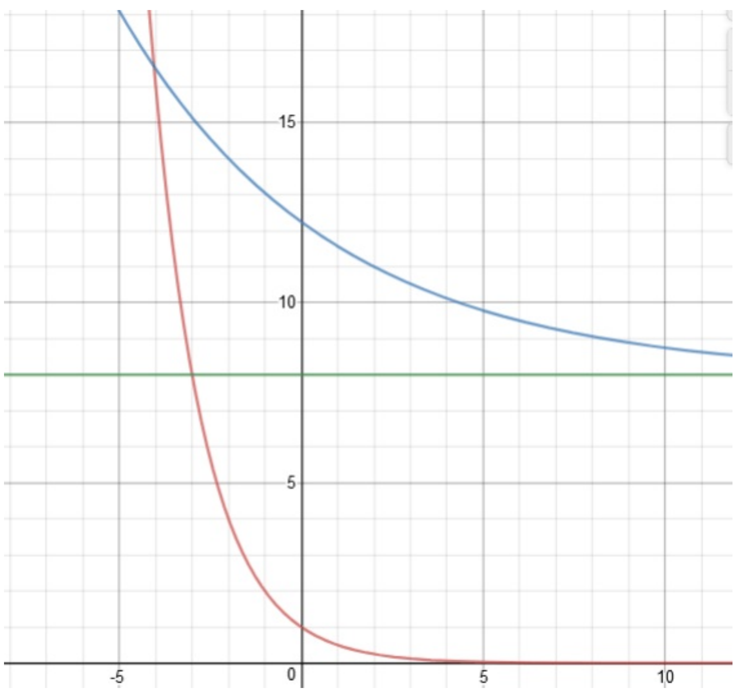
(i) $y=-2\left(\frac{1}{2}\right)^{3(x-1)}+4$

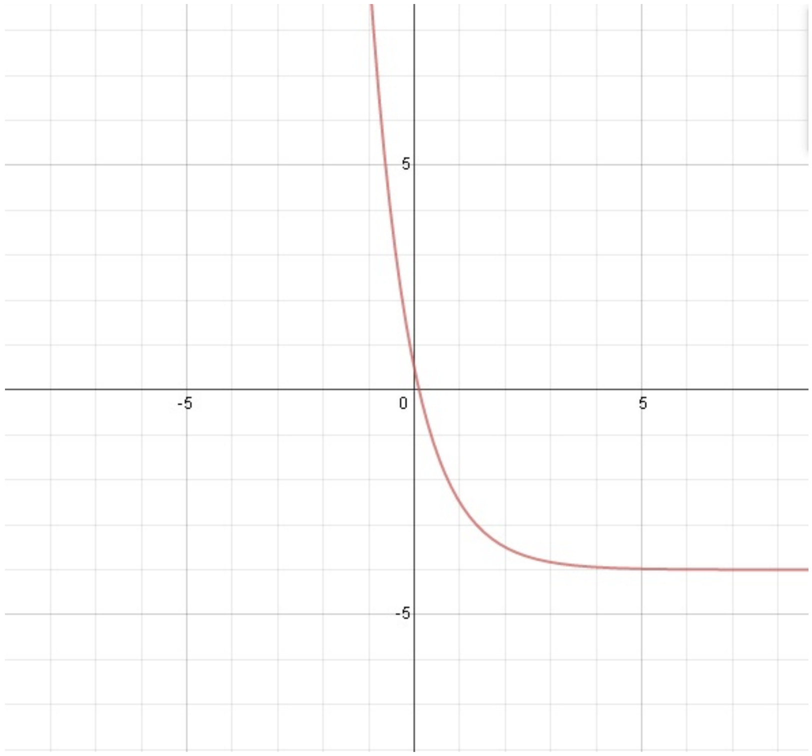
(ii) $y=\frac{1}{2}(3)^{-x+2}-4$

ii)



iii)





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7-3 Solving Exponential Equations (variable is in the exponent)

Method Systematic Trial:

Ex) Solve for x:

$2^x = 10$ Since $2^3 = 8$ $2^4 = 16$

Solution is in between 3 and 4

Test value for x	Power	Approximate value
3.3	$2^{3.3}$	9.849
3.4	$2^{3.4}$	10.556
The value obtained for $2^{3.3}$ is closer to 10, so the next estimate should be closer to 3.3 than to 3.4.		
3.31	$2^{3.31}$	9.918
3.32	$2^{3.32}$	9.987
3.33	$2^{3.33}$	10.056

They should reason that the best estimate is $x \doteq 3.32$ because 9.987 is closer to 10 than 10.056 is.

Method 2 Algebraically DOPublic

Ex) Solve for x algebraically where appropriate:

(i) $9^{2x+1} = 81$

(ii) $16^{2x+1} = \left(\frac{1}{2}\right)^{x-3}$

(iii) $40 = 4^{2x+3}$

(i) $\left(\frac{1}{3}\right)^{2x-1} = (81)^{3-x}$

(ii) $5\left(\frac{1}{4}\right)^{x} = 80$

(iii) $\sqrt{5} = 25^{x-1}$

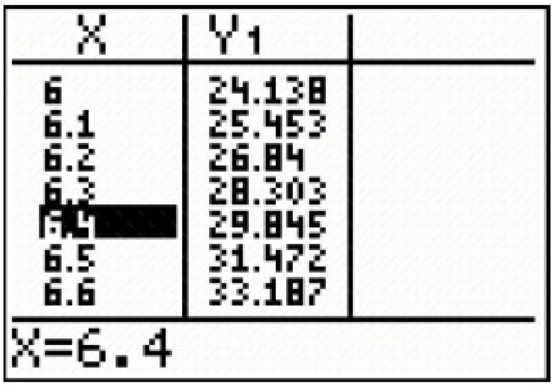
(iv) $27^{2x-1} = \sqrt[3]{3}$

(v) $\sqrt[5]{8^{x-1}} = \sqrt[3]{16^{x}}$

(vi) $\sqrt{3^{x}} = 9^{2x+1}$

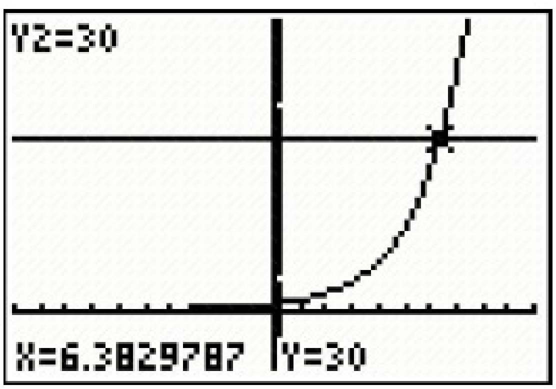
Method III Graphically

Ex) Solve for x: $1.7^x = 30$



X	Y_1
6	24.138
6.1	25.453
6.2	26.84
6.3	28.303
6.4	29.845
6.5	31.472
6.6	33.187

X=6.4



7-3 Applications of Exponential Functions

Exponential Growth:

Model: $y = A_\circ(c)^x$

$A_0 = initial\ amount$ C= 1+r = 1 + growth rate

Ex)

Shelly initially invests \$500 and the value of the investment increases by 4% annually.

- Create a function to model the situation.
- How much money is in Shelly's investment after 30 years?
- What amount of time will it take for the investment to double?

Ex) A house is purchased for \$155,000 and is appreciating at a rate of 2% per year. What will the house be worth in 10 years? How long would it take for the house to double its value?

Model:

in 10 yrs:

Double original value:35 yrs use table on calc

Exponential Growth Cted!

Incremental periods other than 1 year. (It means x goes up by more than 1)

Doubling time $y = A_0(2)^{\frac{t}{d}}$

$A_0 = initial\ amount$ t=any time t d= doubling time

Ex) A bacterial culture doubles in size every 12 hours. If there are 200 bacteria initially present, determine how many will be present 2 days from now.

B) Algebraically, determine how long it would take in days for the culture to reach a population of 819,200?

C) 6,553,600

Ex) After 54 hours, a bacteria culture of 15 cells has 120 cells present. Using an exponential model, determine the doubling time of the bacteria.

Other Incremental periods. Model:

$y = A_0(b)^{\frac{t}{c}}$ $A_0 = initial\ amount$, c = when the object is incrementing

Ex) A trout population will quadruple in size every 8 years. If 1600 trout were initially present, determine algebraically how long it will take the population to reach 409,600.

Ex) A Wayne Gretzky rookie card increases in value 2% every third year. Recently, a card sold for $94,613 at auction. Determine the value of the card 12 years from now.

Ex) Determine an exponential growth function for the data below:

A)

x	0	1	2	3
y	5	6.25	7.8125	

B)

x	-1	0	1	2
y	2	4	8	16

C)

x	0	3	6	9
y	40	60	90	135

Exponential Decay

Model: $y = A_\circ (b)^x$

$A_0 = initial\ amount$ b=1-r (1 minus the growth rate)

Ex) A car depreciates 20% every year. The purchase price was $26,000. Determine the value of the car after 6 years. How long will it take for the car to be worth half of its value?

Applications of Exponential Functions Cted.

Half-life: how long it takes for a radioactive substance to decay to half of its original amount

IE: C14 has a half life of 5730 years. This means if there are 50 mg of C14 present, then it take 5730 years for 50 mg to decay to 25 mg....11,460 years for 50 mg to decay down to 12.5 mg

The half-life of Radon 222 is 92 hours. From an initial sample of 48 g, how long would it take to decay to 6 g?

Model: $y = A_0(\frac{1}{2})^{\frac{t}{h}}$

Half -Life continued April , 2019

The half-life of a certain radioactive isotope is 30 hours. Ask students to algebraically determine the amount of time it takes for a sample of 1792 mg to decay to 56 mg.

$$y = A_0(\frac{1}{2})^{\frac{t}{h}}$$

$56 = 1792(.5)^{(t/30)}$

$1/32 = .5^{(t/30)}$

$2^{-5} = (2^{-1})^{(t/30)}$

$2^{-5} = 2^{-(t/30)}$

$-5 = -t/30$ therefore $t = 5 \times 30 = 150$ hours

Ex) A radioactive element is decaying from 160 mg to 20 mg in 24 hours. Compute the half life of the element.

Model: $A = 160(\frac{1}{2})^{\frac{t}{h}}$ $therefore\ 20 = 160\left(\frac{1}{2}\right)^{\frac{24}{h}}$, $\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{24}{h}}$

Further Applications

If an amount is growing exponentially at certain increments then models similar to doubling time and half-life have to be used.

Page 364: 1- to 4 assigned 5,9,10,11,12

0	1	2	3	4
7	21	63	189	567

Model: y= $7(3)^x$

0	5	10	15	20
7	21	63	189	567

Model: $y = 7(3)^{\frac{x}{5}}$

0	6	12	18	24
80°C	80x.7 = 56	39.2	27.44	19.2

A cup of hot chocolate is served at an initial temperature of 80°C and then allowed to cool in a stadium with an air temperature of 5°C. The difference between the hot chocolate temperature and the temperature of the room will decrease by 30% every 6 minutes. If T represents the temperature of the hot chocolate in degrees Celsius, measured as a function of time, *t*, in minutes, students can answer the following questions:

- What is the transformed exponential function in the form

$$T = a(c)^{b(t-h)} + k?$$

Table:

0	6	12	18	24
80	56	39.2	27.44	19.208

Since the room is 5 degrees it will cool to this temperature therefore this is the asymptote

$$T = 80(.70)^{\frac{t}{6}} + 5$$

- What is the temperature at time $t = 11$ minutes?

- How long does it take the hot chocolate to cool to a temperature of 40°C?

Print Page on e from Guide for Students

Financial Applications (New for this course)

Compound Interest: Earning interest on interest paid every period.

Compounding Periods are (Could be):
Yearly
semi-annually
Monthly 12 times
Semi-Monthly 24 times a year
Bi-weekly vs 26 times
Weekly 52
Daily 365

Tyrell has $3000 to invest. The JB Bank offers to pay 7% per year compounded semi-annually for three years. The TH Bank offers 6% per year compounded monthly for three years. Ask students to:

(i) Write functions to model each bank's offer.

(ii) Use the functions to determine which bank will give Tyrell the maximum return on his investment.

Model: Model:

Ask students to choose between two investment options and justify their choice: earning 12% interest per year compounded annually or 12% interest per year compounded monthly.

Use a $10,000 investment for 5 years.

Model Model

Determine how long $1100 needs to be invested in an account that earns 7% compounded semi-annually before it increases in value to $1500.

The Number "e"

e

The number **e** is a famous irrational number, and is one of the most important numbers in mathematics.

The first few digits are:

2.7182818284590452353602874713527 (and more ...)

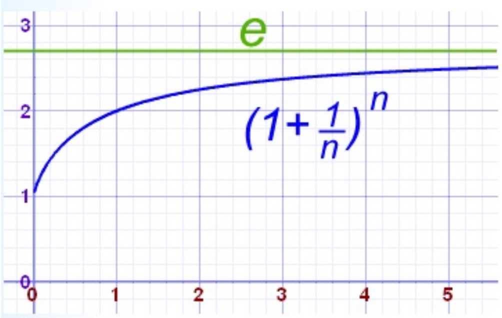
*It is often called **Euler's number** after Leonhard Euler (pronounced "Oiler").*

e is the base of the Natural Logarithms (invented by John Napier).

There are many ways of calculating the value of **e**, but none of them ever give a totally exact answer, because **e** is irrational (not the ratio of two integers).

But it **is** known to over 1 trillion digits of accuracy!

For example, the value of $(1 + 1/n)^n$ approaches **e** as n gets bigger and bigger:



n	$(1 + 1/n)^n$
1	2.00000
2	2.25000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827

Another Calculation

The value of e is also equal to $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$ (etc)

(Note: "!" means factorial)

The first few terms add up to: $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.718055556$

- Ask students to evaluate:

  (i) π^2

  (ii) e^2

  (iii) $2e$

(RF9.3)

- Ask students to estimate a solution to the following equations using benchmarks:

  (i) $e^x = 200$

  (ii) $e^{x+1} = 1000$

(RF9.2, RF9.3)

- Ask students to sketch the graph of $y = e^x$, identify the y-intercept, state the equation of the horizontal asymptote, and state the domain and range.

(RF8.1, RF8.2)

- Ask students to graph $y = 2^x$, $y = e^x$, and $y = 3^x$ using graphing technology. Ask students to compare the domains, ranges, y - intercepts and equations of the horizontal asymptotes of the graphs.

(RF8.1, RF8.2)

End