Unit 6
Calculus of Trigonometry

In this unit, students will evaluate limits involving trigonometric expressions and determine the derivative of expressions involving trigonometric functions. They will revisit the Chain Rule and implicit differentiation.

Students will be introduced to inverse trigonometric functions, determine the exact value of an expression involving an inverse trigonometic function, and solve problems involving the derivative of an inverse trigonometric function.

Achievement Indicator:

C10.1 *Establish each of the following trigonometric limits, using informal methods:*

- $\lim_{x \to 0} \sin x = 0$
- $\lim_{x \to 0} \cos x = 1$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

In other words, for any real number *a*,

$$\lim_{\theta \to a} \sin \theta = \sin a \qquad \lim_{\theta \to a} \cos \theta = \cos a$$

t the

θ).

This enables us to evaluate certain limits quite simply. For example,

$$\lim_{x \to \pi} x \cos x = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cdot \cos \pi = -\pi$$

Functions with the Direct Substitution Property are called *continuous at a*.

0 (1, 0) x

Figure 1

As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$.

Thus

$$\boxed{1} \qquad \lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

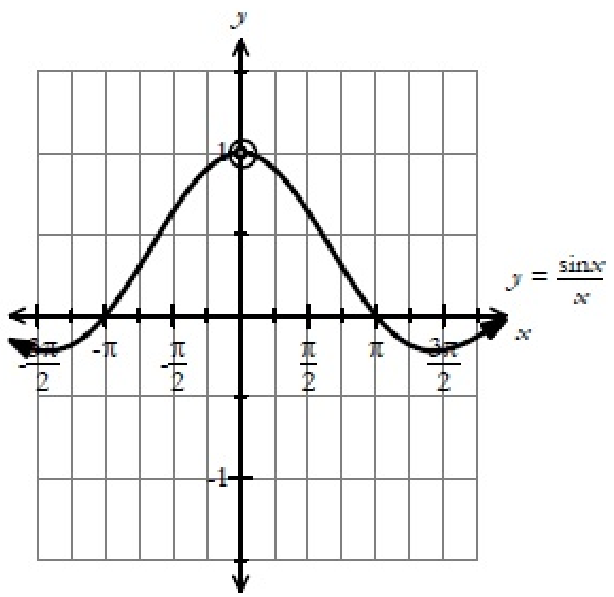
Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in $\boxed{1}$ assert that the cosine and sine functions satisfy the Direct Substitution Property at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions satisfy the Direct Substitution Property everywhere.

Special Limits with Trig Expressions (must study and know!)

1) $$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad why?$$

Provide students with the graph of $y = \frac{\sin x}{x}$ and ask them to examine the value of y for values of x close to 0. It may help students to organize their information in a table.

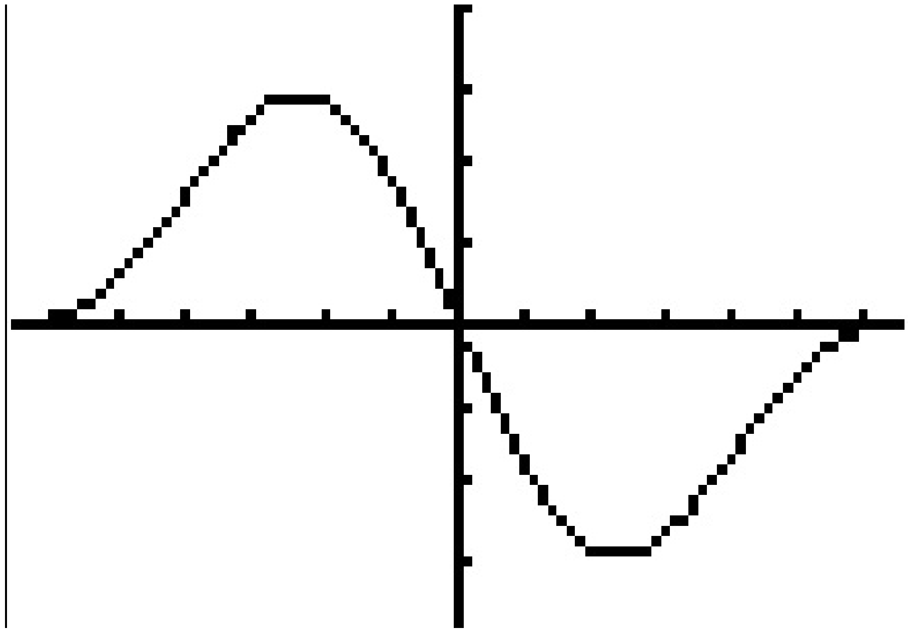


x	y
-2	0.45
-1	0.84
-0.1	0.998
-0.01	0.9999
0	
0.01	0.9999
0.1	0.998
1	0.84
2	0.45

Note: direct sub yields 0 over 0 form

2)

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \quad why?$$



X	Y1
-.06	.02999
-.05	.02499
-.04	.02
-.03	.015
-.02	.01
-.01	.005
0	ERROR

X	Y1
.06	-.03
.05	-.025
.04	-.02
.03	-.015
.02	-.01
.01	-.005
0	ERROR

Evaluate $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$.

Solution:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right)$$

$$= \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)}$$

$$= -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right)$$

$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$

$$= -1 \cdot \left(\frac{0}{1 + 1} \right)$$ (by Equation 6)

$$= 0$$

Evaluating Limits Using Trig Expressions

C10.2 *Evaluate limits involving trigonometric expressions.*

Evaluate: $\dfrac{\lim}{x \to 0} \dfrac{\sin 5x}{x}$ 0/0 form rewrite as:

The whole idea is to try and get it in the form of : see *

$$\lim_{x\to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5}$$

$$\lim_{x\to 0} \frac{\sin 5x}{5x} \cdot 5$$

Let $u = 5x$

As $x \to 0$, then $5x \to 0$

then

$$\lim_{u\to 0} \frac{\sin u}{u} \cdot 5$$

$$5 \lim_{u\to 0} \frac{\sin u}{u}$$

$$5(1)$$

$$5$$

In general (KNOW!)
*

This can lead into a discussion around the general case $\lim_{x\to 0} \frac{\sin kx}{x} = k$, where $k \neq 0$.

Ex) Evaluate:

$\lim_{x\to 0} \frac{\sin 7x}{6x}$ as $x \to 0$, then $7x \to 0$ 0/0 form let u = 7x

$\lim_{u\to 0} \frac{\sin u}{\frac{6u}{7}}$

$\frac{7}{6} \lim_{u\to 0} \frac{\sin u}{u}$

$\frac{7}{6}(1)$

$\frac{7}{6}$

Evaluate A) $\frac{\lim}{t \to} \frac{\sin 3t}{2t}$ B) $\frac{\lim}{t \to 0} \frac{\sin(-4t)}{t}$ C) $\frac{\lim}{t \to 0} \frac{\sin^2 5t}{t^2}$

There are a number of strategies that can be used to evaluate trigonometric limits: simplifying, factoring, rationalizing, and rewriting the trigonometric expression. As students evaluate a limit such as $\lim_{x\to 0}\frac{\sin 2x}{2x^2+x}$, they should first notice the limit as an indeterminate form. Ask students to factor the denominator $\lim_{x\to 0}\frac{\sin 2x}{x(2x+1)}$ and rewrite the limit as $\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{1}{(2x+1)}$. They should then observe that since the $\lim_{x\to 0}\frac{\sin 2x}{x} = 2$ and $\lim_{x\to 0}\frac{1}{(2x+1)} = 1$, the value of $\lim_{x\to 0}\frac{\sin 2x}{2x^2+x}$ is 2.

remember as well tan x =,

- Ask students to evaluate the following:

(i)	$\lim_{x\to 0}\frac{\sin 5x}{3x}$	(ii)	$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}$
(iii)	$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}$	(iv)	$\lim_{t\to 0}\frac{\tan 6t}{t}$
(v)	$\lim_{\theta\to 0}\frac{\sin 3\theta \sin 5\theta}{\theta^2}$	(vi)	$\lim_{x\to 0}(x+\sin x)$
(vii)	$\lim_{x\to \frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}$	(viii)	$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$

(C10.2)

- Ask students to determine the vertical asymptotes of $f(x) = \tan x$. They should desribe the behaviour of $f(x)$ to the left and right of each vertical asymptote.

Ex) Evaluate the $\lim_{x \to 0} \frac{\sin 2x}{2x^2 + x}$

Page 160

Derivatives of Trig Functions

The Sine and Cosine Functions

Consider the following two limits:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

By using the above two limits we have:

4 $$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

5 $$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$f'(x) = \sin x \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$f'(x) = \sin x(0) + \cos x(1)$$

$$f'(x) = \cos x$$

Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, we can use the Quotient Rule to find the derivative of the tangent function:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

Trigonometric Functions

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

We collect all the differentiation formulas for trigonometric functions in the following table.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Remember that they are valid only when *x* is measured in radians.

Example 7

Differentiate: $y = 3 \sin \theta + 4 \cos \theta$.

Solution:

$$\frac{dy}{d\theta} = 3 \frac{d}{d\theta} (\sin \theta) + 4 \frac{d}{d\theta} (\cos \theta)$$

$$= 3 \cos \theta - 4 \sin \theta$$

The equation of the tangent line at x = π would be:

The chain rule with the trig functions. (Write out!)

Errors generally occur when more than one application of the derivative is applied. Initially, students may confuse the format of the expression. They may incorrectly write $\sin(5x + 1)^4$ as $[\sin(5x + 1)]^4$. Ensure they recognize that $\sin^4(5x + 1)$ is equivalent to$[\sin(5x + 1)]^4$. Ask them to differentiate the expressions and discuss the similarities and differences in their solutions.

$$\frac{d}{dx}\sin^4(5x+1) = 4\sin^3(5x+1)\cdot\cos(5x+1)\cdot 5$$

whereas

$$\frac{d}{dx}\sin(5x+1)^4 = \cos(5x+1)^4\cdot 4(5x+1)^3\cdot 5$$

Remind students to apply the Product Rule in situations where they have xy. Students should differentiate an expression such as

$\frac{d}{dx}\sin(xy) = \cos(xy)\cdot(xy' + y)$.

Ex) find the equation of the tangent line to the curve

$$y = \frac{\cos(3x)}{\sin(2x)} \ at\ x = \frac{\pi}{3}.$$

Page 112 3,4,8,9,10, 20, 21, 22, 23, 34, 35, 29

- Ask students to determine $\frac{dy}{dx}$ for $x\cos(xy) = y\sin(3x)$.

(C11.2)

- Ask students to differentiate $y = \cos x$ repeatedly six times. They should explain the pattern they observe and use it to determine the 24^{th} derivative of $y = \cos x$.

y = cos x $\quad y' = -\sin x,\ y'' = -\cos x,\ y''' = \sin x,\ y^{iv} = \cos x$

Ask students to determine the equation of the tangent line of the relation $\sin(xy - y^2) = x^2 - 1$ at the point (1,1).

(C11

- Ask students to determine the derivatives of the following:
  - (i) $f(x) = \frac{\sin x}{x^2}$
  - (ii) $f(x) = 3\sin x + \cos x - 1$
  - (iii) $f(x) = (\sec x)(\csc x)$
  - (iv) $y = \tan\sqrt{1-x}$

Optimization with Trig Functions (Max and Min=extrema)

- The position of a particle as it moves horizontally is described by the equation $s = 2\sin t - \cos t$, $0 \leq t \leq 2\pi$, where s is displacement in metres and t is the time in seconds. Find the absolute maximum and minimum displacements.

$$v = \frac{ds}{dt} = 2\cos t - (-\sin t)$$

$$v = \frac{ds}{dt} = 2\cos t + \sin t$$

critical numbers

$$2\cos t + \sin t = 0$$

$$\sin t = -2\cos t$$

$$\frac{\sin t}{\cos t} = -2$$

$$\tan t = -2$$

negative in Q 2 and Q 4

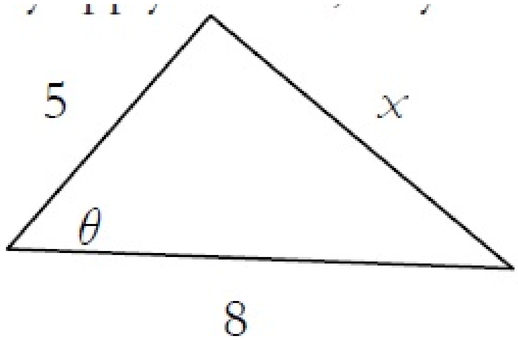
For $\frac{\pi}{2} < t < \pi$, $t_1 = \pi - 1.107 = 2.0$ sec

For $\frac{3\pi}{2} < t < 2\pi$, $t_2 = 2\pi - 1.107 = 5.2$ sec

radian mode!

Related Rates and Trig Functions

Two sides of a triangle measure 5 m and 8 m in length. The angle between them is increasing at a rate of $\frac{\pi}{45} \frac{rad}{\sec}$. How fast is the length of the third side changing when the contained angle is $\frac{\pi}{3}$?



$$x^2 = 5^2 + 8^2 - 2(5)(8)\cos\theta$$

$$x^2 = 89 - 80\cos\theta$$

Differentiate both sides wrt to time

$$2x\frac{dx}{dt} = -80(-\sin\theta)\cdot\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{40\sin\theta}{x}\cdot\frac{d\theta}{dt}$$

Given: $\frac{d\theta}{dt} = \frac{\pi}{45}\frac{rad}{\sec}, \theta = \frac{\pi}{3}$

$$x^2 = 89 - 80\cos(\tfrac{\pi}{3})$$

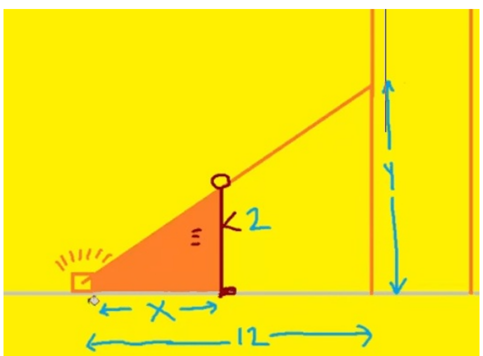
$$x = 7m$$

Therefore, $\frac{dx}{dt} = \frac{40\sin(\frac{\pi}{3})}{7}\cdot\frac{\pi}{45}$

$$\frac{dx}{dt} = 0.35\tfrac{m}{s}$$

https://www.youtube.com/watch?v=uDkWSsy2Y2M

A spotlight on the ground shines on a wall 12 metres away. If a man 2 metres tall walks from the spotlight toward the building at a speed of 1.6 metres per second, how fast is his shadow on the building shrinking when he is 4 metres from the building?

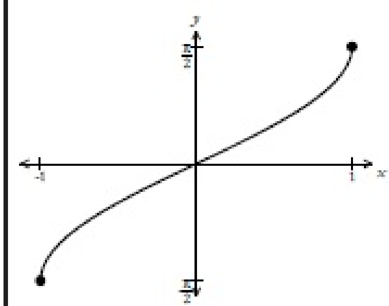
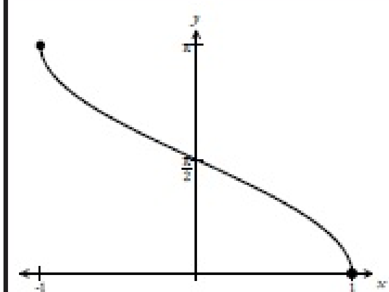
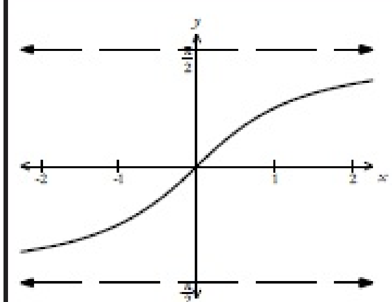


The position of a particle as it moves horizontally is described by the equation $s = 2\sin t + \sin 2t$, $-\pi \leq t \leq \pi$. If s is the displacement in metres and t is the time in seconds, determine the absolute maximum and absolute minimum displacements.

Inverse Trigonometry

C12.1 *Explain the relationship between the primary trigonometric functions and the inverse trigonometric functions.*

C12.2 *Explain why trigonometric functions have their domains restricted to create inverse trigonometric functions.*

Inverse Trigonometric Function	Domain	Range	Graph
$\sin^{-1}x = \theta$ iff $\sin\theta = x$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	[-1,1]	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	
$\cos^{-1}x = \theta$ iff $\cos\theta = x$ $0 \le \theta \le \pi$	[-1,1]	$[0, \pi]$	
$\tan^{-1}x = \theta$ iff $\tan\theta = x$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	

Students should be able to distinguish between reciprocal trigonometric functions (secant, cosecant, cotangent) and inverse trigonometric functions. They should be aware, for example, that $\frac{1}{\sin x} \neq \sin^{-1} x$. The "-1" is not an exponent, it is a notation that is used to denote inverse trigonometric functions. Point out to students that if the "-1" represented an exponent it would be written as $(\sin(x))^{-1} = \frac{1}{\sin x}$. Expose students to another notation for inverse trigonometric functions, $\sin^{-1}x = \arcsin(x)$, that avoids this ambiguity.

- Ask students to sketch the graphs of the inverse trigonometric functions $y = \sin^{-1}x$, $y = \cos^{-1}x$ and $y = \tan^{-1}x$. They should then use graphing technology to verify their results. As teachers observe students' work, ensure they discuss the following characteristics:

  (i) graph of $\sin^{-1}x$ is a reflection of the graph of $\sin x$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) about the line $y = x$

  (ii) graph of $\cos^{-1}x$ is a reflection of the graph of $\cos x$ (restricted to $[0, \pi]$) about the line $y = x$

  (iii) graph of $\tan^{-1}x$ is a reflection of the graph of $\tan x$ (restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$) about the line $y = x$

  (iv) the domain of a function and the range of its inverse are the same

  (v) the vertical asymptotes of the tangent function become horizontal asymptotes for the inverse function

Ex) Evaluate: $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ same as $\arcsin(\frac{\sqrt{3}}{2})$

What is this asking? It is asking you to find an angle within the restriction $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ for y = arcsin x $= \sin^{-1} x$?

$\sin x = \frac{\sqrt{3}}{2}$ Q1 only because of restriction

therefore x= $60° = \frac{\pi}{3}$

Ex) Determine the exact value of

$A)\ \tan^{-1}(-1) \quad B)\ arc\cos(-.5) \quad C)\ \sin^{-1}(-2) \quad D)\ \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Page 297 1 to 7

Ex) Find:

$A)\ \sin^{-1}(\sin\frac{7\pi}{4})\ B)\ \cos(\cos^{-1}\frac{\sqrt{3}}{2})\ C)\ \sin^{-1}(\cos\frac{7\pi}{3})\ D)\ \sin^{-1}(\sin\frac{\pi}{6})\ E)\ \sin^{-1}(\sin\frac{2\pi}{3})$

In general: Arcsin (sin x)= x when
Arccos (cos x)= x when
Arctan (tan x) = x when

$\sin\left(\sin^{-1} x\right) = x\ for$

$\cos(\cos^{-1} x) = x\ for$

$\tan(\tan^{-1} x) = x\ for$

$\sin^{-1}(\ x\ \ ) = \pi$

$\tan^{-1}(\ 1\ \ ) = x$

$Cos^{-1}(\quad) =$

$\sin(\ \ x\ )\ \ = r$

$Cos(\ \ y\ ) = x$

- Ask students to answer the following:

  (i) Given $\theta = \sin^{-1}(\frac{8}{17})$, find the values of all six trigonometric functions at θ.

  (ii) If the point (-3, 4) is on the terminal arm of θ, find the values of all six trigonometric functions at θ.

  (iii) Evaluate $\tan(\cos^{-1}(\frac{7}{11}))$.

  (iv) Find the exact value of $\sin(2\tan^{-1}\sqrt{2})$.

  (v) Find the exact value of $\cos(\tan^{-1}2 + \tan^{-1}3)$.

C12.5 *Derive the inverse trigonometric derivatives.*

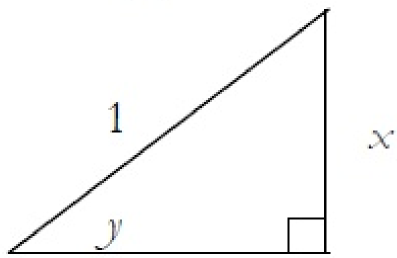
- differentiate $x = \sin y$ implicitly

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\sec y = \frac{dy}{dx}$$

- construct a reference triangle to record the relationship between x and y where $\sin y = \frac{x}{1}$. Remind students that since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, the reference triangle can be drawn in quadrant I or IV. Students should, however, result in the same answer. The length of the hypotenuse is always positive.



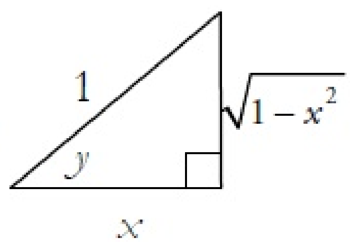
- ask students to determine an expression for the missing side of the right triangle.

- use substitution to determine the ratio for sec *y*: $\sec y = \frac{1}{\sqrt{1-x^2}}$

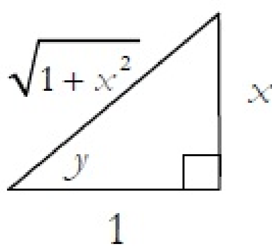
- write an expression for the derivative: $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Why?

- If $y = \cos^{-1}x$, then $x = \cos y$ 

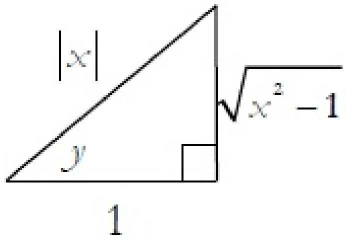
where $0 \le y \le \pi$

- If $y = \tan^{-1}x$, then $x = \tan y$ 

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

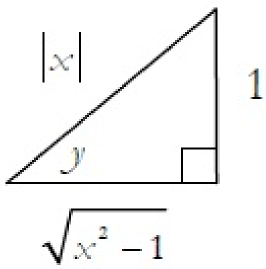
- differentiate both sides of the equation resulting in $\frac{dy}{dx} = \frac{1}{\cos y}$
- use substitution where $\cos^2 y = 1 - \sin^2 y$ and $x = \sin y$ to conclude that $\cos y = \sqrt{1-x^2}$.
- Rewrite as $\frac{dy}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

- If $y = \sec^{-1}x$, then $x = \sec y$



where $0 \le y < \frac{\pi}{2}$ or $\pi \le y < \frac{3\pi}{2}$

- If $y = \csc^{-1}x$, then $x = \csc y$



where $0 < y \le \frac{\pi}{2}$ or $\pi < y \le \frac{3\pi}{2}$

Teachers should draw attention to the derivatives of $y = \sec^{-1}x$ and $y = \csc^{-1}x$ that involve $|x|$. Remind students that x represents the length of the hypotenuse, therefore, its measure is always positive.

At this point, it is expected that most problems are composite in nature. Ask students to differentiate expressions similar to $\sin^{-1}(1 - x^2)$, $\tan^{-1}(\sin x)$ and $x^2\cos^{-1}\left(\frac{2}{x}\right)$.

Calculus 3208

Evaluating Derivatives of Inverse Trig Functions

Ex)

Find the derivative of $y = \cos^{-1} 5x$.

Put $u = 5x$ so $y = \cos^{-1} u$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}\\ &= \frac{-1}{\sqrt{1-(5x)^2}}\frac{d(5x)}{dx}\\ &= \frac{-5}{\sqrt{1-25x^2}}\end{aligned}$$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Inverse Trigonometric Functions

Each of these formulas can be combined with the Chain Rule. For instance, if u is a differentiable function of x, then

$$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \text{and} \qquad \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx}$$

Differentiate $f(x) = x\tan^{-1}\sqrt{x}$.

Solution:

$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}\frac{1}{2}x^{-1/2} + \tan^{-1}\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \tan^{-1}\sqrt{x}$$

Ex 3

Find the derivative of $y = 3\cos^{-1}(x^2 + 0.5)$.

Put $u = x^2 + 0.5$.

Then we have: $y = 3\ \cos^{-1} u$

$$y = 3\ \cos^{-1}\left(x^2 + 0.5\right)$$

$$\frac{dy}{dx} = 3\left[\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}\right]$$

$$= 3\left[\frac{-1}{\sqrt{1-\left(x^2+0.5\right)^2}}\frac{d\left(x^2+0.5\right)}{dx}\right]$$

$$= 3\left[\frac{-1}{\sqrt{1-\left(x^2+0.5\right)^2}}(2x)\right]$$

$$= \frac{-6x}{\sqrt{1-\left(x^2+0.5\right)^2}}$$

Expanding and simplifying gives:

$$\frac{dy}{dx} = \frac{-6x}{\sqrt{0.75 - x^4 - x^2}}$$

Ex 4

Find the derivative of $y = (x^2 + 1)\sin^{-1} 4x$.

$$y = \left(x^2 + 1\right)\sin^{-1} 4x$$

Let $u = x^2 + 1$ and $v = \sin^{-1} 4x$.

Then

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$= \left(x^2 + 1\right)\left[\frac{1}{\sqrt{1 - (4x)^2}}\,(4)\right] + \left(\sin^{-1} 4x\right)(2x)$$

$$= \frac{4\left(x^2 + 1\right)}{\sqrt{1 - 16x^2}} + 2x\ \sin^{-1} 4x$$

Exercises are on the next page to practice: Do not write should be on print out on a sheet!

- Ask students to identify and correct any errors in the given solution.

$$y = \tan^{-1}(3x)$$
$$\frac{dy}{dx} = \frac{1}{1+(3x)^2}$$
$$\frac{dy}{dx} = \frac{1}{1+9x^2}$$

(C12.6)

- Ask students to find $f'(x)$ for each of the following:

  (i) $f(x) = \arcsin x + \arccos x$

  (ii) $f(x) = \cos^{-1}\sqrt{2x-1}$

  (iii) $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

  (iv) $f(x) = (\tan^{-1} x)^{-1}$

  (v) $f(x) = \cos^{-1}\left(\dfrac{b + a\cos x}{a + b\cos x}\right)$

(C12.6)

3 Find dy/dx

i) $y = \sin^{-1}(1 - x^2)$ *ii*) $y = \tan^{-1}(\sin x)$ *iii*) $y = x^2 \cos^{-1}(\frac{2}{x})$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

- Ask students to differentiate $y^2 \sin x = \tan^{-1}(x) - y$ with respect to x.

- Ask students to write the equation of the tangent line to:

  (i) $f(x) = x\sin^{-1}(\frac{x}{4}) + \sqrt{16 - x^2}$ at $x = 2$.

  (ii) $y = \arccos(\frac{x}{2})$ at the point $(1, \pi)$.

  (iii) $y = \tan^{-1}x$ at the point $(1, \frac{\pi}{4})$.

- A particle moves horizontally so that its displacement in metres after t seconds is given by $s(t) = \tan^{-1}(\sqrt{t-1})$. Ask students to determine the velocity of the particle when $t = 10$ sec.

End of Unit!