Mathematics

Advanced Mathematics 3200

Interim Edition

Newfoundland Labrador

Curriculum Guide
2013

Contents

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INTRODUCTION

Background

The curriculum guide communicates high expectations for students.

The Mathematics curriculum guides for Newfoundland and Labrador have been derived from *The Common Curriculum Framework for 10-12 Mathematics: Western and Northern Canadian Protocol*, January 2008. These guides incorporate the conceptual framework for Grades 10 to 12 Mathematics and the general outcomes, specific outcomes and achievement indicators established in the common curriculum framework. They also include suggestions for teaching and learning, suggested assessment strategies, and an identification of the associated resource match between the curriculum and authorized, as well as recommended, resource materials.

Beliefs About Students and Mathematics

Mathematical understanding is fostered when students build on their own experiences and prior knowledge.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in developing mathematical literacy is making connections to these backgrounds and experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students, and enhance within them the formation of sound, transferable mathematical understandings. Students at all levels benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions provide essential links among concrete, pictorial and symbolic representations of mathematical concepts.

The learning environment should value and respect the diversity of students' experiences and ways of thinking, so that students feel comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. They must come to understand that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable.

Affective Domain

To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

A positive attitude is an important aspect of the affective domain and has a profound impact on learning. Environments that create a sense of belonging, encourage risk taking and provide opportunities for success help develop and maintain positive attitudes and self-confidence within students. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting, asssessing and revising personal goals.

Goals For Students

Mathematics education must prepare students to use mathematics confidently to solve problems.

The main goals of mathematics education are to prepare students to:

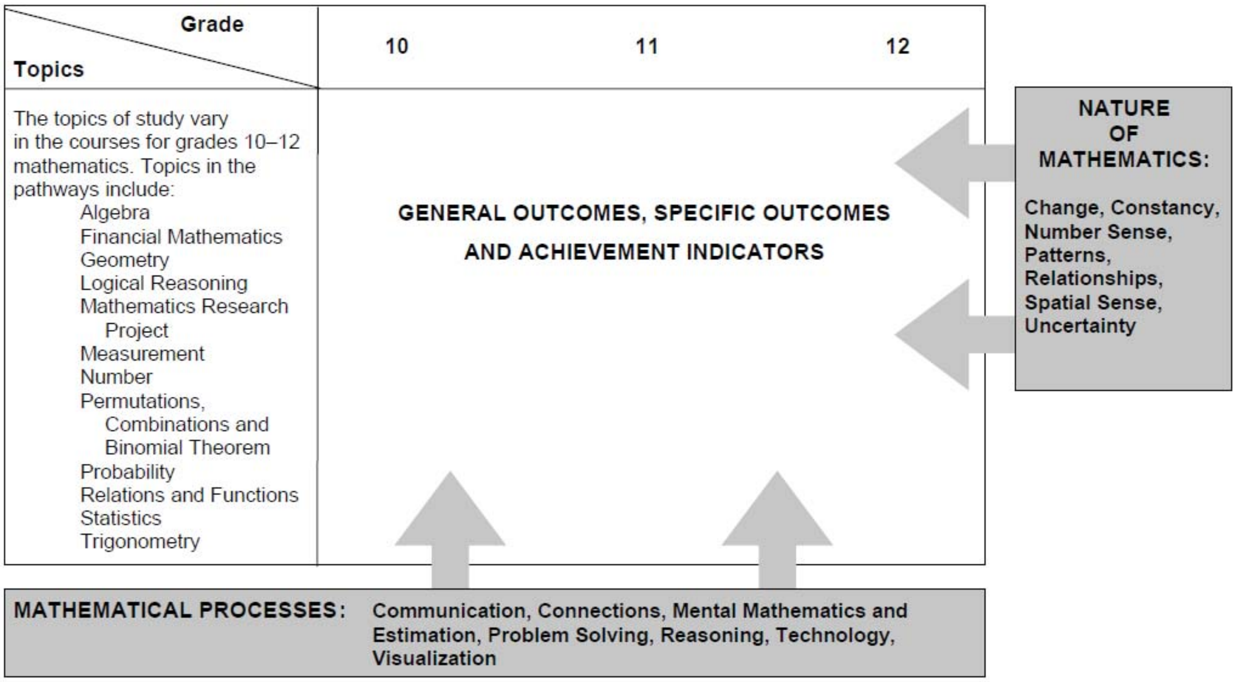
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity.

CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



Mathematical Processes

- *Communication [C]*
- *Connections [CN]*
- *Mental Mathematics and Estimation [ME]*
- *Problem Solving [PS]*
- *Reasoning [R]*
- *Technology [T]*
- *Visualization [V]*

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and embrace lifelong learning in mathematics.
Students are expected to:

- communicate in order to learn and express their understanding
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technologies as tools for learning and for solving problems
- develop visualization skills to assist in processing information, making connections and solving problems.

This curriculum guide incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students must be able to communicate mathematical ideas in a variety of ways and contexts.

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication helps students make connections among concrete, pictorial, symbolic, oral, written and mental representations of mathematical ideas.

Connections [CN]

Through connections, students begin to view mathematics as useful and relevant.

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged.

The brain is constantly looking for and making connections. “Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding … Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching” (Caine and Caine, 1991, p.5).

Mental Mathematics and Estimation [ME]

Mental mathematics and estimation are fundamental components of number sense.

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "... become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving" (Rubenstein, 2001, p. 442).

Mental mathematics "... provides the cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers" (Hope, 1988, p. v).

Estimation is used for determining approximate values or quantities or for determining the reasonableness of calculated values. It often uses benchmarks or referents. Students need to know when to estimate, how to estimate and what strategy to use.

Estimation assists individuals in making mathematical judgements and in developing useful, efficient strategies for dealing with situations in daily life.

Problem Solving [PS]

Learning through problem solving should be the focus of mathematics at all grade levels.

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, "How would you know?" or "How could you ...?", the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

A problem-solving activity requires students to determine a way to get from what is known to what is unknown. If students have already been given steps to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly seek and engage in a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk takers.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics.

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics.

Mathematical experiences in and out of the classroom provide opportunities for students to develop their ability to reason. Students can explore and record results, analyze observations, make and test generalizations from patterns, and reach new conclusions by building upon what is already known or assumed to be true.

Reasoning skills allow students to use a logical process to analyze a problem, reach a conclusion and justify or defend that conclusion.

Technology [T]

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

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Technology can be used to:

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- decrease the time spent on computations when other mathematical learning is the focus
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- create geometric patterns
- simulate situations
- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and which estimation strategies to use (Shaw and Cliatt, 1989).

Nature of Mathematics

- *Change*
- *Constancy*
- *Number Sense*
- *Patterns*
- *Relationships*
- *Spatial Sense*
- *Uncertainty*

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this curiculum guide. The components are change, constancy, number sense, patterns, relationships, spatial sense and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics.

Change is an integral part of mathematics and the learning of mathematics.

Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as:

- the number of a specific colour of beads in each row of a beaded design
- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Constancy

Constancy is described by the terms stability, conservation, equilibrium, steady state and symmetry.

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS-Benchmarks, 1993, p.270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The ratio of the circumference of a teepee to its diameter is the same regardless of the length of the teepee poles.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Number Sense

An intuition about number is the most important foundation numeracy.

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p.146).

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Mastery of number facts is expected to be attained by students as they develop their number sense. This mastery allows for facility with more complex computations but should not be attained at the expense of an understanding of number.

Number sense develops when students connect numbers to their own real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. The evolving number sense typically comes as a by product of learning rather than through direct instruction. It can be developed by providing rich mathematical tasks that allow students to make connections to their own expereinces and their previous learning.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns.

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all strands of mathematics.

Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with, and understanding of, their environment.

Patterns may be represented in concrete, visual or symbolic form. Students should develop fluency in moving from one representation to another.

Students must learn to recognize, extend, create and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems.

Learning to work with patterns in the early grades helps students develop algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships.

Mathematics is one way to describe interconnectedness in a holistic worldview. Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects and concepts. The search for possible relationships involves collecting and analyzing data and describing relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense offers a way to interpret and reflect on the physical environment.

Spatial sense involves visualization, mental imagery and spatial reasoning. These skills are central to the understanding of mathematics.

Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 3-D objects and 2-D shapes and to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of shapes and objects. Spatial sense allows students to make predictions about the results of changing these dimensions; e.g., doubling the length of the side of a square increases the area by a factor of four. Ultimately, spatial sense enables students to communicate about shapes and objects and to create their own representations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty.

Uncertainty is an inherent part of making predictions.

The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

Essential Graduation Learnings

Essential graduation learnings are statements describing the knowledge, skills and attitudes expected of all students who graduate from high school. Essential graduation learnings are cross-curricular in nature and comprise different areas of learning: *aesthetic expression, citizenship, communication, personal development, problem solving, technological competence* and *spiritual and moral development*.

Aesthetic Expression

Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship

Graduates will be able to assess social, cultural, economic and environmental interdependence in a local and global context.

Communication

Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) and mathematical and scientific concepts and symbols to think, learn and communicate effectively.

Personal Development

Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving

Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language and mathematical and scientific concepts.

Technological Competence

Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.

Spiritual and Moral Development

Graduates will be able to demonstrate an understanding and appreciation for the place of belief systems in shaping the development of moral values and ethical conduct.

See *Foundations for the Atlantic Canada Mathematics Curriculum*, pages 4-6.

The mathematics curriculum is designed to make a significant contribution towards students' meeting each of the essential graduation learnings (EGLs), with the communication, problem-solving and technological competence EGLs relating particularly well to the mathematical processes.

Outcomes and Achievement Indicators

The curriculum is stated in terms of general outcomes, specific outcomes and achievement indicators.

General Outcomes

General outcomes are overarching statements about what students are expected to learn in each course.

Specific Outcomes

Specific outcomes are statements that identify the specific skills, understanding and knowledge that students are required to attain by the end of a given course.
In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for illustrative purposes or clarification, and are not requirements that must be addressed to fully meet the learning outcome.

Achievement Indicators

Achievement indicators are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome.

Specific curriculum outcomes represent the means by which students work toward accomplishing the general curriculum outcomes and ultimately, the essential graduation learnings.

Program Organization

<table>
<tr><th>Program Level</th><th>Course 1</th><th>Course 2</th><th>Course 3</th><th>Course 4</th></tr>
<tr><td>Advanced</td><td rowspan="2">Mathematics 1201</td><td>Mathematics 2200</td><td>Mathematics 3200</td><td>Mathematics 3208</td></tr>
<tr><td>Academic</td><td>Mathematics 2201</td><td>Mathematics 3201</td><td></td></tr>
<tr><td>Applied</td><td>Mathematics 1202</td><td>Mathematics 2202</td><td>Mathematics 3202</td><td></td></tr>
</table>

The applied program is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the workforce.

The academic and advanced programs are designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs. Students who complete the advanced program will be better prepared for programs that require the study of calculus.

The programs aim to prepare students to make connections between mathematics and its applications and to become numerate adults, using mathematics to contribute to society.

Summary

The conceptual framework for Grades 10-12 Mathematics (p. 3) describes the nature of mathematics, mathematical processes and the mathematical concepts to be addressed. The components are not meant to stand alone. Activities that take place in the mathematics classroom should result from a problem-solving approach, be based on mathematical processes and lead students to an understanding of the nature of mathematics through specific knowledge, skills and attitudes among and between topics.

ASSESSMENT AND EVALUATION

Purposes of Assessment

What learning is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others about what is really valued.

Assessment techniques are used to gather information for evaluation. Information gathered through assessment helps teachers determine students' strengths and needs in their achievement of mathematics and guides future instructional approaches.

Teachers are encouraged to be flexible in assessing the learning success of all students and to seek diverse ways in which students might demonstrate what they know and are able to do.

Evaluation involves the weighing of the assessment information against a standard in order to make an evaluation or judgment about student achievement.

Assessment has three interrelated purposes:

- assessment *for* learning to guide and inform instruction;
- assessment *as* learning to involve students in self-assessment and setting goals for their own learning; and
- assessment *of* learning to make judgements about student performance in relation to curriculum outcomes.

Assessment *for* Learning

Assessment *for* learning involves frequent, interactive assessments designed to make student understanding visible. This enables teachers to identify learning needs and adjust teaching accordingly. It is an ongoing process of teaching and learning.

Assessment *for* learning:

- requires the collection of data from a range of assessments as investigative tools to find out as mush as possible about what students know
- provides descriptive, specific and instructive feedback to students and parents regarding the next stage of learning
- actively engages students in their own learning as they assess themselves and understand how to improve performance.

Assessment *as* Learning

Assessment *as* learning actively involves students' reflection on their learning and monitoring of their own progress. It focuses on the role of the student as the critical connector between assessment and learning, thereby developing and supporting metacognition in students.

Assessment *as* learning:

- supports students in critically analysing their learning related to learning outcomes
- prompts students to consider how they can continue to improve their learning
- enables students to use information gathered to make adaptations to their learning processes and to develop new understandings.

Assessment *of* Learning

Assessment *of* learning involves strategies to confirm what students know, demonstrate whether or not they have met curriculum outcomes, or to certify proficiency and make decisions about students' future learning needs. Assessment *of* learning occurs at the end of a learning experience that contributes directly to reported results.

Traditionally, teachers relied on this type of assessment to make judgments about student performance by measuring learning after the fact and then reporting it to others. Used in conjunction with the other assessment processes previously outlined, however, assessment *of* learning is strengthened.

Assessment *of* learning:

- provides opportunities to report evidence to date of student achievement in relation to learning outcomes, to parents/guardians and other stakeholders
- confirms what students know and can do
- occurs at the end of a learning experience using a variety of tools.

Because the consequences of assessment *of* learning are often far-reaching, teachers have the responsibility of reporting student learning accurately and fairly, based on evidence obtained from a variety of contexts and applications.

Assessment Strategies

Assessment techniques should match the style of learning and instruction employed. Several options are suggested in this curriculum guide from which teachers may choose, depending on the curriculum outcomes, the class and school/district policies.

Observation (formal or informal)

This technique provides a way of gathering information fairly quickly while a lesson is in progress. When used formally, the student(s) would be aware of the observation and the criteria being assessed. Informally, it could be a frequent, but brief, check on a given criterion. Observation may offer information about the participation level of a student for a given task, use of a concrete model or application of a given process. The results may be recorded in the form of checklists, rating scales or brief written notes. It is important to plan in order that specific criteria are identified, suitable recording forms are ready, and all students are observed within a reasonable period of time.

Performance

This curriculum encourages learning through active participation. Many of the curriculum outcomes promote skills and their applications. In order for students to appreciate the importance of skill development, it is important that assessment provide feedback on the various skills. These may be the correct manner in which to use a manipulative, the ability to interpret and follow instructions, or to research, organize and present information. Assessing performance is most often achieved through observing the process.

Paper and Pencil

These techniques can be formative or summative. Whether as part of learning, or a final statement, students should know the expectations for the exercise and how it will be assessed. Written assignments and tests can be used to assess knowledge, understanding and application of concepts. They are less successful at assessing processes and attitudes. The purpose of the assessment should determine what form of paper and pencil exercise is used.

Journal

Journals provide an opportunity for students to express thoughts and ideas in a reflective way. By recording feelings, perceptions of success, and responses to new concepts, a student may be helped to identify his or her most effective learning style. Knowing how to learn in an effective way is powerful information. Journal entries also give indicators of developing attitudes to mathematical concepts, processes and skills, and how these may be applied in the context of society. Self-assessment, through a journal, permits a student to consider strengths and weaknesses, attitudes, interests and new ideas. Developing patterns may help in career decisions and choices of further study.

Interview

This curriculum promotes understanding and applying mathematics concepts. Interviewing a student allows the teacher to confirm that learning has taken place beyond simple factual recall. Discussion allows a student to display an ability to use information and clarify understanding. Interviews may be a brief discussion between teacher and student or they may be more extensive. Such conferences allow students to be proactive in displaying understanding. It is helpful for students to know which criteria will be used to assess formal interviews. This assessment technique provides an opportunity to students whose verbal presentation skills are stronger than their written skills.

Presentation

The curriculum includes outcomes that require students to analyze and interpret information, to be able to work in teams, and to communicate information. These activities are best displayed and assessed through presentations. These can be given orally, in written/pictorial form, by project summary, or by using electronic systems such as video or computer software. Whatever the level of complexity, or format used, it is important to consider the curriculum outcomes as a guide to assessing the presentation. The outcomes indicate the process, concepts and context for which a presentation is made.

Portfolio

Portfolios offer another option for assessing student progress in meeting curriculum outcomes over a more extended period of time. This form of assessment allows the student to be central to the process. There are decisions about the portfolio, and its contents, which can be made by the student. What is placed in the portfolio, the criteria for selection, how the portfolio is used, how and where it is stored, and how it is evaluated are some of the questions to consider when planning to collect and display student work in this way. The portfolio should provide a long-term record of growth in learning and skills. This record of growth is important for individual reflection and self-assessment, but it is also important to share with others. For all students, it is exciting to review a portfolio and see the record of development over time.

INSTRUCTIONAL FOCUS

Planning for Instruction

Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- By decreasing emphasis on rote calculation, drill and practice, and the size of numbers used in paper and pencil calculations, more time is available for concept development.
- Problem solving, reasoning and connections are vital to increasing mathematical fluency and must be integrated throughout the program.
- There should be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using manipulatives and be developed concretely, pictorially and symbolically.
- Students bring a diversity of learning styles and cultural backgrounds to the classroom. They will be at varying developmental stages.

Teaching Sequence

The curriculum guide for Advanced Mathematics 3200 is organized by units. This is only a suggested teaching order for the course. There are a number of combinations of sequences that would be appropriate.

Each two page spread lists the topic, general outcome, and specific outcome.

Instruction Time Per Unit

The suggested number of hours of instruction per unit is listed in the guide at the beginning of each unit. The number of suggested hours includes time for completing assessment activities, reviewing and evaluating. The timelines at the beginning of each unit are provided to assist in planning. The use of these timelines is not mandatory. However, it is mandatory that all outcomes are taught during the school year, so a long term plan is advised. Teaching of the outcomes is ongoing, and may be revisited as necessary.

Resources

The authorized resource for Newfoundland and Labrador students and teachers is *Pre-Calculus 12* (McGraw-Hill Ryerson). Column four of the curriculum guide references *Pre-Calculus 12* for this reason.
Teachers may use any other resource, or combination of resources, to meet the required specific outcomes.

GENERAL AND SPECIFIC OUTCOMES

GENERAL AND SPECIFIC OUTCOMES WITH ACHIEVEMENT INDICATORS (pages 19-206)

This section presents general and specific outcomes with corresponding achievement indicators and is organized by unit. The list of indicators contained in this section is not intended to be exhaustive but rather to provide teachers with examples of evidence of understanding that may be used to determine whether or not students have achieved a given specific outcome. Teachers may use any number of these indicators or choose to use other indicators as evidence that the desired learning has been achieved. Achievement indicators should also help teachers form a clear picture of the intent and scope of each specific outcome.

Advanced Mathematics 3200 is organized into nine units: *Polynomial Functions, Function Transformations*, *Radical Functions*, *Trigonometry and the Unit Circle, Trigonometric Functions and Graphs, Trigonometric Identities*, *Exponential Functions, Logarithmic Functions,* and *Permutations, Combinations and the Binomial Theorem.*

Polynomial Functions

Suggested Time: 14 Hours

Unit Overview

Focus and Context

Previous work with quadratic functions will be extended in this unit to include polynomial functions of degree ≤ 5.

Students will apply the integral zero theorem, synthetic division, and factoring techniques to determine the zeros of cubic, quartic and quintic functions. They will relate the zeros to the *x*-intercepts of the graph and then graph and analyze the polynomial functions.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF11

Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

SCO RF10

Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

Mathematical Processes

[C] Communication
[CN] Connections
[ME] Mental Mathematics and Estimation
[PS] Problem Solving
[R] Reasoning
[T] Technology
[V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions		
AN5. Demonstrate an understanding of common factoring and trinomial factoring, concretely, pictorially and symbolically. [C, CN, R, V] RF1. Interpret and explain the relationships among data, graphs and situations. [C, CN, R, T, V] RF2. Demonstrate an understanding of relations and functions. [C, R, V] RF6. Relate linear relations expressed in: • slope-intercept form $y = \mathrm{m}x + \mathrm{b}$ • general form $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ • slope-point form $y - y_1 = \mathrm{m}(x - x_1)$ to their graphs. [CN, R, T, V]	RF1. Factor polynomial expressions of the form: • $ax^2 + bx + c,\ a \neq 0$ • $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$ • $a(f(x))^2 + b(f(x)) + c,\ a \neq 0$ • $a^2(f(x))^2 - b^2(g(y))^2,\ a \neq 0,\ b \neq 0$ where a, b and c are rational numbers. [CN, ME, R] RF3. Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: • vertex • domain and range • direction of opening • axis of symmetry • x- and y-intercepts. [CN, R, T, V] RF4. Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: • vertex • domain and range • direction of opening • axis of symmetry • x- and y-intercepts and to solve problems. [CN, PS, R, T, V] RF5. Solve problems that involve quadratic equations. [C, CN, PS, R, T, V] RF7. Solve problems that involve linear and quadratic inequalities in two variables. [C, PS, T, V] RF8. Solve problems that involve quadratic inequalities in one variable. [CN, PS, V]	RF10. Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME] RF11. Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]

Relations and Functions

Outcomes

Students will be expected to

RF11 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

[C, CN, T, V]

Achievement Indicators:

RF11.1 *Identify the polynomial functions in a set of functions, and explain the reasoning.*

RF11.2 *Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students related linear relations expressed in slope-intercept form, general form and slope-point form to their graphs (RF6). In Mathematics 2200, they analyzed quadratic functions to identify the characteristics of the corresponding graph (RF3, RF4). In this unit, students graph and analyze polynomial functions of degree 5 or less.

Linear and quadratic functions are examples of polynomial functions that students have already studied. They will now extend their study of polynomials to include cubic, quartic and quintic functions. A function of the form $a_nx^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$, is a polynomial of degree n, and the number a_n is the leading coefficient. Students have previously sketched the graph of polynomial functions of degree 0, 1 and 2. They should recognize the following:

Function	Degree	Type of Function	Graph
$f(x) = a$	0	constant	horizontal line
$f(x) = ax + b$	1	linear	line with slope a
$f(x) = ax^2 + bx + c$	2	quadratic	parabola

They will now be introduced to some of the basic features of the graphs of polynomial functions of degree greater than 2:

- The graph of a polynomial function is continuous.
- The graph of a polynomial function has only smooth turns. A function of degree n has at most $n - 1$ turns.
- If the leading coefficient of the polynomial function is positive, then the graph rises to the right. If the leading coefficient is negative, then the graph falls to the right.
- The constant term is the y-intercept of the graph.

The intent at this point is that students learn to recognize these basic features. Later in the unit, they will use these features, point-plotting, and intercepts to make reasonably accurate sketches.

To examine the basic features, students could graph polynomials with technology, such as graphing calculators, or other graphing software, such as FX Draw and WinPlot. Graphing software apps available for students' mobile devices could also be utilized.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Students should determine whether or not each of the following is a polynomial function and explain the reasoning.

  (i) $f(x) = x^2 - \sqrt{x}$

  (ii) $f(x) = 3x^3 + 4$

(RF11.1)

- Ask students to identify the features of the graph related to the function $f(x) = -3x^2 + 9x + x^5$.

(RF11.2)

Interview

- Ask students to answer the following:
  (i) How many turns can the graph of a polynomial function of degree 5 have? Explain.
  (ii) Describe the characteristics of the graphs of cubic and quartic functions with the largest possible number of terms.

(RF11.2)

Performance

- Students can work in groups to match the equations with the appropriate graph. They should explain the features that guided the selection of the appropriate graph.

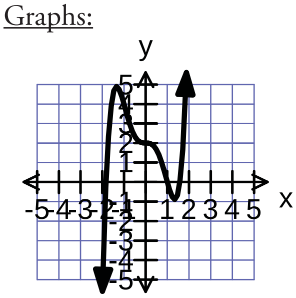
Functions:

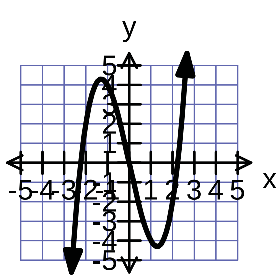
$f(x) = x^3 - 5x$

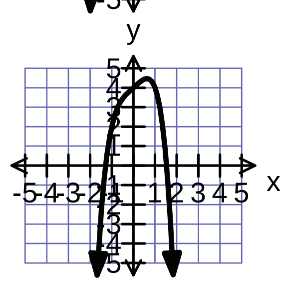
$f(x) = x^5 - 3x^3 + 2$

$f(x) = -x^4 + x + 4$

Graphs:







(RF11.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.1 Characteristics of Polynomial Functions

Student Book (SB): pp. 106-117

Teacher's Resource (TR): pp. 62-66

Relations and Functions

Outcomes

Students will be expected to

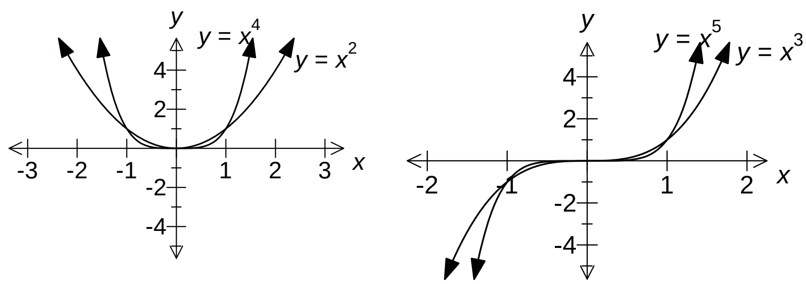
RF11 Continued ...

Achievement Indicator:

RF11.3 *Generalize rules for graphing polynomial functions of odd or even degree.*

Elaborations—Strategies for Learning and Teaching

Students should explore the graphs of various polynomials with even degree and odd degree. The polynomial functions that have the simplest graphs are the monomial functions $f(x) = a_n x^n$. When n is even, the graph is similar to the graph of $f(x) = x^2$. When n is odd, the graph is similar to the graph of $f(x) = x^3$. The greater the value of n, the flatter the graph of a monomial is on the interval $-1 \leq x \leq 1$.



Through exploration, students should see that if the degree of a polynomial function is even, then its graph has the same behaviour to the left and right. The graph of $f(x) = x^4$, for example, rises to the right and rises to the left. It extends up into Quadrant II and up into Quadrant I. If the degree is odd, the graph has opposite behaviours to the right and left. The graph of $f(x) = -x^3$, for example, falls to the right and rises to the left. It extends up into Quadrant II and down into Quadrant IV.

Students could use graphing technology to determine any similarities and differences between polynomials such as the following:

- $f(x) = 2x + 1$
- $f(x) = x^2 + 2x - 3$
- $f(x) = x^3 + 2x^2 - x - 2$
- $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$
- $f(x) = 0.2x^5 - x^4 - 2x^3 + 10x^2 + 1.4x - 9$

They could then graph each of these with a negative leading coefficient. From this, they should identify a pattern in the graphs of odd and even degree functions. They should note patterns in the end behaviour, the constant term and the number of real x-intercepts. Examples should be limited to polynomials with real x-intercepts to allow students to easily identify the patterns.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to create a foldable or a graphic organizer to summarize the rules for graphing polynomial functions of odd or even degree.

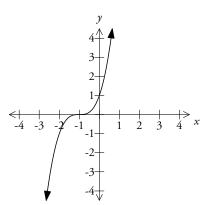
(RF11.3)

Performance

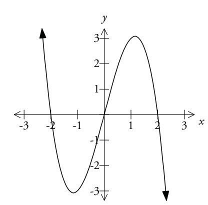
- The activity *Commit and Toss* gives students an opportunity to anonymously commit to an answer and provide a justification for the answer they selected. Provide students with a selected response question, as shown below. Students write their answer, crumble their solutions into a ball, and toss the papers into a basket. Once all papers are in the basket, ask students to reach in and take one out. They then move to the corner of the room designated to match the selected response on the paper they have taken. In their respective corners, they should discuss the similarities and differences in the explanation provided and report back to the class.

Which of the following is the graph of an even degree function?

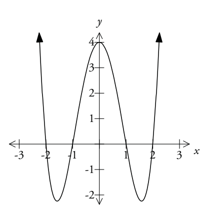
(A)



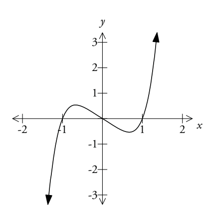
(B)



(C)



(D)



Explain your reasoning:

(RF11.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.1 Characteristics of Polynomial Functions

SB: pp. 106-117

TR: pp. 62-66

Relations and Functions

Outcomes

Students will be expected to

RF10 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5 with integral coefficients).

[C, CN, ME]

Achievement Indicators:

RF10.1 *Explain how long division of a polynomial expression by a binomial expression of the form x - a, a∈ I, is related to synthetic division.*

RF10.2 *Divide a polynomial expression by a binomial expression of the form x - a, a∈ I using long division or synthetic division.*

Elaborations—Strategies for Learning and Teaching

Students were introduced to factoring techniques for quadratic functions in Mathematics 1201 (AN5). They used common factors and trinomial factoring to express quadratics in factored form. In Mathematics 2200, these factoring techniques were extended (RF1) to factor expressions with rational coefficients of the forms:

- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
- $a(f(x))^2 + b(f(x)) + c,\ a \neq 0$
- $a^2(f(x))^2 - b^2(g(y))^2,\ a \neq 0,\ b \neq 0.$

In this unit, these techniques are used to factor cubic, quartic and quintic polynomials.

To factor higher order polynomials, long division or synthetic division is used in combination with factoring techniques. Students should be introduced to synthetic division in terms of it's connection to long division. They should observe that long division with polynomials is similar to division with numerical expressions. To determine the quotient for the expression $(x^3 - 2x^2 + 3x - 4) \div (x + 2)$, for example, either long division or synthetic division could be used. Students should see the connection between the divisor and dividend in long division and the root and coefficients in synthetic division. As a result, synthetic division requires fewer calculations.

Long Division:

$$x+2\overline{\big)\,x^3-2x^2+3x-4}$$

Synthetic Division:

$$\begin{array}{r|rrrr} \underline{-2} & 1 & -2 & 3 & -4 \\ & & -2 & 8 & -22 \\ \hline & 1 & -4 & 11 & \underline{|-26} \end{array}$$

Teachers should work through both processes to allow students to make the connection between each type of division. Synthetic division can be regarded as a more efficient method than doing long division of two polynomials when the divisor is a linear function of the form $(x - a)$. Although synthetic division can be completed using either the addition or subtraction operation, it is recommended that the addition operation be used. This helps make a connection between synthetic division and the remainder theorem.

Students should see that there may be a remainder at the end of the process. In the example above,

$$\left(x^3-2x^2+3x-4\right)\div\left(x+2\right)=\left(x^2-4x+11\right)+\frac{-26}{x+2}.$$

Students should also be encouraged to include any restrictions that may exist. In this particular case, $x \neq -2$. They determined non-permissible values for rational expressions in Mathematics 2200 (AN4).

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Ask students to explain the connection between long division and synthetic division, using the example $(2x^3 - x^2 - 13x - 6) \div (x + 2)$.

(RF10.1)

Paper and Pencil

- Ask students to write an incorrect, but plausible, solution for the division statement $(x^4 - x^3 - 8x^2 + 8) \div (x + 2)$. They should then exchange solutions with a partner and find the errors in the solution.

(RF10.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.2 The Remainder Theorem

SB: pp. 118-125

TR: pp. 67-70

Relations and Functions

Outcomes

Students will be expected to

RF10 Continued ...

Achievement Indicators:

RF10.2 *Continued*

RF10.3 *Explain the relationship between the remainder when a polynomial expression is divided by x - a, a ∈ I, and the value of the polynomial expression at x = a (remainder theorem).*

RF10.4 *Explain and apply the factor theorem to express a polynomial expression as a product of factors.*

RF10.5 *Explain the relationship between the linear factors of a polynomial expression and the zeros of the corresponding polynomial function.*

Elaborations—Strategies for Learning and Teaching

Encourage students to carefully examine the polynomial when setting up synthetic division. A zero must be included for each missing term in the dividend. A common error occurs in questions such as $\left(x^4-10x^2+2x+3\right)\div\left(x-3\right)$ when students do not include zero for the missing cubic term. Students should check the result by multiplying the quotient and the divisor. This can help inform them if the division is done correctly.

Students should be introduced to the remainder theorem which says that if $(x - a)$ is a linear divisor of a polynomial function $p(x)$, then $p(a)$ is the remainder. Initially, students may not see a need for this theorem since they can obtain remainders by using synthetic division. Exposure to polynomials such as $x^9 - 1$ or $x^{100} + 1$ should help them see that the remainder theorem is more efficient in some cases.

The factor theorem, which states that a polynomial, $P(x)$, has a factor $x - a$ if and only if $P(a) = 0$, can be used to determine the factors of a polynomial expression. Students use the integral zero theorem to relate the factors of a polynomial and the constant term of the polynomial. The factors of the constant term indicate possible factors of the polynomial. They then verify using the factor theorem. For the polynomial $f\left(x\right)=x^3-3x^2-4x+12$, for example, the possible integral zeros are the factors of 12. Remind students to test both the positive and negative factors. Since $f(2) = 0$, $f(-2) = 0$ and $f(3) = 0$, the factors of the polynomial are $(x - 2)$, $(x + 2)$ and $(x - 3)$. This method is restricted to polynomial functions with distinct integral zeros only.

Students should work with polynomial functions that also have non-integer zeros. The factor theorem can also be used in conjunction with synthetic division to factor a polynomial. Students should use the integral zero theorem to determine possible factors and verify one of the factors using the factor theorem. Synthetic division is then applied, resulting in a polynomial to be factored further. For the polynomial $f\left(x\right)=4x^3-12x^2+5x+6$, $f(2) = 0$. Using synthetic division gives:

$$\begin{array}{r|rrrr} 2 & 4 & -12 & 5 & 6 \\ & & 8 & -8 & -6 \\ \hline & 4 & -4 & -3 & \boxed{0} \end{array}$$

This results in $(x - 2)(4x^2 - 4x - 3)$, which can be further factored to $(x -2)(2x - 3)(2x + 1)$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the remainder for the function $f(x) = x^9 + 3x^4 - 5x + 1$ if it is divided by $x - 3$.

(RF10.3)

- The volume of a triangular prism is given by $V = x^3 + 18x^2 + 80x + 96$. Ask students to determine the missing dimension(s) if two of the dimensions are $(x + 2)$ and $(x + 12)$. They should identify any restrictions on the variable x.

(RF10.4)

Interview

- Ask students to explain why the remainder theorem is an efficient way to find the remainder when a polynomial expression is divided by $x - a$.

(RF10.3)

Performance

- Ask students to explain how to determine the remainder when $10x^4 - 11x^3 - 8x^2 + 7x + 9$ is divided by $2x - 3$.

(RF10.3)

- Ask students to explain and demonstrate, using the integral zero theorem, the factor theorem and synthetic division, how they would determine the values of k that make $(x - k)$ a factor of $f(x) = x^3 - 4x^2 - 11x + 30$.

(RF10.4, RF10.2)

- Ask students to create a polynomial of degree ≥ 3 using linear factors. They should exchange with other students and explain the relationship between linear factors and the corresponding zeros of the function.

(RF10.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.2 The Remainder Theorem

SB: pp. 118-125

TR: pp. 67-70

3.3 The Factor Theorem

SB: pp. 126-135

TR: pp. 71-74

Relations and Functions

Outcomes

Students will be expected to

RF10 Continued ...

Achievement Indicators:

RF10.4, RF10.5 *Continued*

Elaborations—Strategies for Learning and Teaching

Students should also be able to use alternative methods to factor polynomials, such as grouping. They should have been introduced to grouping when they factored quadratics using decomposition. This can be extended to polynomials such as $f(x) = x^3 - 3x^2 - 4x + 12$:

$$f(x) = x^3 - 3x^2 - 4x + 12$$
$$f(x) = x^2(x-3) - 4(x-3)$$
$$f(x) = (x-3)(x^2-4)$$
$$f(x) = (x-3)(x+2)(x-2)$$

Once a polynomial has been factored, students apply the zero product property to determine the zeros. This is a natural extension of the work done with solving quadratic equations in Mathematics 2200.

RF11 Continued ...

Students were introduced to the basic features of the graphs of polynomial functions of degree greater than 2 at the beginning of this unit. They will now use these features, along with the intercepts, to graph polynomials of degree 5 or less.

RF11.4 *Explain the relationship among the following:*

- *the zeros of a polynomial function*
- *the roots of the corresponding polynomial equation*
- *the x-intercepts of the graph of the polynomial function.*

Students were introduced to the zeros of a quadratic function, the roots of the quadratic equation, and the x-intercepts of the graph in Mathematics 2200 (RF5). It is important that they distinguish between the terms zeros, roots and x-intercepts, and use the correct terms in a given situation. Students could be asked to find the roots of the equation $3x^3 - 10x^2 - 23x - 10 = 0$, find the zeros of the function $f(x) = 3x^3 - 10x^2 - 23x - 10$, or determine the x-intercepts of the graph of $f(x) = 3x^3 - 10x^2 - 23x - 10$. In each case, they are identifying the factors of the polynomial and solving to arrive at the solution $x = -1$, $x = -\frac{2}{3}$, $x = 5$.

Students should realize that the degree of a polynomial indicates the maximum number of x-intercepts for its graph. For each real x-intercept there is a linear factor and a zero for the polynomial function.

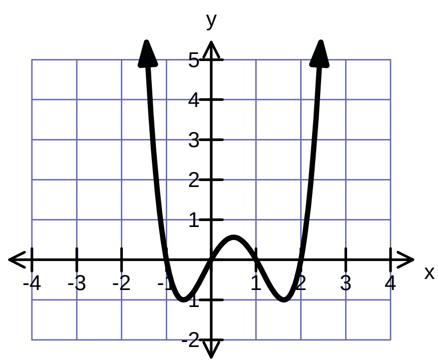
General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Given the equation and the graph below, ask students to identify the zeros, roots and x-intercepts and explain how they are determined.

  $f(x) = x(x - 1)(x + 1)(x - 2)$



(RF11.4)

- Ask students to answer the following questions for the polynomial $f(x) = x^3 - 4x^2 + x + 6$.

  (i) Algebraically or graphically determine the linear factors of $f(x)$.

  (ii) Explain the relationship between the linear factors and any x-intercepts.

(RF11.4, RF10.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.3 The Factor Theorem

SB: pp. 126-135

TR: pp. 71-74

3.4 Equations and Graphs of Polynomial Functions

SB: pp. 136-152

TR: pp. 75-79

Relations and Functions

Outcomes

Students will be expected to

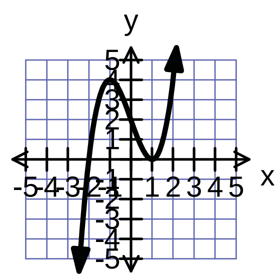
RF11 Continued ...

Achievement Indicator:

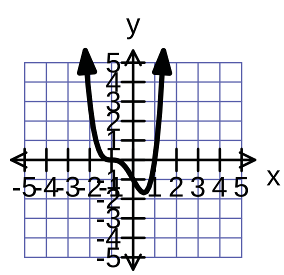
RF11.5 *Explain how the multiplicity of a zero of a polynomial function affects the graph.*

Elaborations—Strategies for Learning and Teaching

Students have solved polynomials to find distinct, real zeros that correspond to distinct, real x-intercepts. They need to be aware that the zeros will not always be distinct. Some polynomial functions may have a multiplicity of a zero (i.e., double, triple, etc.), also referred to as the order of a zero. Students should graph polynomials such as $f(x) = (x - 1)(x - 1)(x + 2)$ or $f(x) = (x + 1)(x + 1)(x + 1)(x - 1)$to see the effect of multiplicity of roots.



$f(x) = (x - 1)(x - 1)(x + 2)$



$f(x) = (x + 1)(x + 1)(x + 1)(x - 1)$

Ask students questions such as:

- What would happen to the graph if there was a zero of multiplicity 4?
- What would be the effect on the graph if there are two double roots?
- What effect does a triple root have on the graph? a double root? a single root?

Students should be encouraged to check other possibilities for multiplicity of zeros for polynomials of degree ≤ 5 and the effect on their respective graphs.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

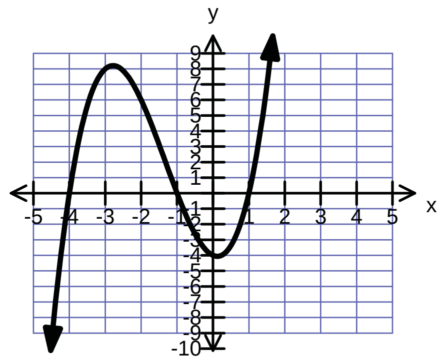
Interview

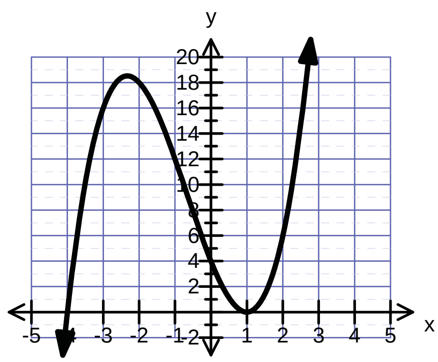
- Ask students to explain, using an example, how a zero of multiplicity 5 affects the graph of a polynomial.

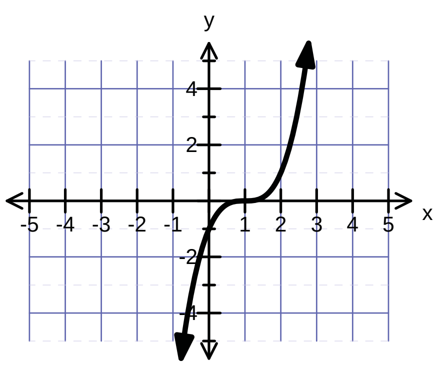
(RF11.5)

Performance

- Ask students to examine the following graphs for the possibility of multiplicity of zeros.







Using *Think-Pair-Share*, give individual students time to think about the similarities and differences among the graphs with respect to multiplicity of zeros. Students then pair up with a partner to discuss their ideas. After pairs discuss, students share their ideas in a small-group or whole-class discussion.

(RF11.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.4 Equations and Graphs of Polynomial Functions

SB: pp. 136-152

TR: pp. 75-79

Relations and Functions

Outcomes

Students will be expected to

RF11 Continued ...

Achievement Indicator:

RF11.6 *Sketch, with or without technology, the graph of a polynomial function.*

Elaborations—Strategies for Learning and Teaching

Students should realize that in order to sketch a graph of any polynomial without graphing technology they have to identify points such as the x- and y-intercepts. They use the value of the leading coefficient to determine the end behaviour and consider how multiplicity of zeros affects the graph of the function.

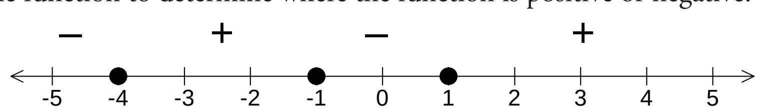
Students should also be able to identify when the function is positive and when it is negative. A table of intervals or a sign diagram consisting of a number line, roots and test points can help with this. They should realize that the function is neither positive nor negative at the x-intercepts. The intervals should be expressed as set or interval notation. Students should be familiar with both types of notation from Mathematics 1201 (RF1).

Ask students to determine the intervals where the graph represented by the function $f(x) = (x + 4)(x + 1)(x - 1)$ is positive or negative.

They could use a table to determine the intervals:

Interval	$x < -4$	$-4 < x < -1$	$-1 < x < 1$	$x > 1$
Sign	–	+	–	+

Sign lines were introduced in Mathematics 2200 when students graphed absolute value functions (RF2). Relating the sign diagram to the x-axis of the graph, students can substitute an x-value from each interval into the function to determine where the function is positive or negative.



Remind students of the effect of a multiplicity of a zero on the graph of a polynomial function. Discuss how this would appear on a sign diagram. Students should realize that for a zero of odd multiplicity (e.g., a single root or a triple root), the sign of the function changes. If a function has a zero of even multiplicity (e.g., a double root), the sign does not change.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Without using graphing technology, students could sketch the graph of a polynomial such as $f(x) = x^3 - 2x^2 - 4x + 8$. They should be exposed to examples where the x-intercepts can be determined using grouping or other factoring techniques.

(RF11.6)

Observation

- Observe as students sketch the graph of $f(x) = -(x + 2)^3(x - 4)$ and verify using graphing technology. Ask them to explain what characteristics of the polynomial (e.g., intercepts, multiplicity of zeros, leading coefficient, positive or negative intervals) helped them sketch the graph.

(RF11.6)

Performance

- *Give Me Five* is an activity that provides students with an opportunity to individually and publicly reflect on their learning. Ask students: What was the most significant thing you learned about graphing polynomials?

  Give time for students to quietly reflect before asking for five volunteers to share their reflections. A show of hands can also indicate how many students had a similar thought each time a student shares his or her reflection.

(RF11.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.4 Equations and Graphs of Polynomial Functions

SB: pp. 136-152

TR: pp. 75-79

Note

SB: pp. 143-144

Students are not expected to use transformations to graph polynomial functions.

Relations and Functions

Outcomes

Students will be expected to

RF11 Continued ...

Achievement Indicator:

RF11.7 *Solve a problem by modeling a given situation with a polynomial function.*

RF11.8 *Determine the equation of a polynomial function given its graph.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students solved problems by determining and analyzing a quadratic equation (RF5). This will now be extended to polynomial functions. When solving a problem, it is often necessary to simplify the problem and express the problem with mathematical language or symbols to make a mathematical model. It is an expectation that students will apply skills developed in this unit to solve problems in various contexts. For example, they should answer questions based on area, volume and numbers, such as:

- Three consecutive integers have a product of −720. What are the integers?

  Students could model this situation with a polynomial function and solve the equation to determine the integers.

- An open box is to be made from a 10-in. by 12-in. piece of cardboard by cutting x-in. squares from each corner and folding up the sides. If the volume of the box is 72 in.^{3}, what are the dimensions?

  It is necessary here to place restrictions on the independent variable. Ask students why, in this case, the value of x is restricted to $0 < x < 5$.

It is important for students to consider the possibility of inadmissable roots in the context of the problem. They should realize that time, length, width and height, for example, cannot be negative values.

Students should also be able to analyze a graph and create a polynomial equation that models the graph. They should be given a variety of graphs that have both distinct roots and multiplicity of roots and asked to determine the equation of the polynomial function that represents the graph. In cases where a graph may possibly represent a polynomial of either degree 3 or 5, or of either degree 2 or 4, students would need to be provided with the degree of the resulting equation.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Students can work in groups of two for the activity *Pass the Problem*. Each pair gets a problem that involves a situation to be modelled with a polynomial function. Ask one student to write the first line of the solution and then pass it to the second student. The second student verifies the workings and checks for errors. If there is an error, students should discuss what the error is and why it occurred. The student then writes the second line of the solution and passes it to the partner. This process continues until the solution is complete.

  Sample Problem:

  The length, width, and height of a rectangular box are x cm, $(x-4)$ cm, and $(x+5)$ cm, respectively. Find the dimensions of the box if the volume is 132 cm^3.

(RF12.7)

Paper and Pencil

- Ask students to answer the following:

  (i) An open-topped box with a volume of 900 cm^3 is made from a rectangular piece of cardboard by cutting equal squares from four corners and folding up the sides.

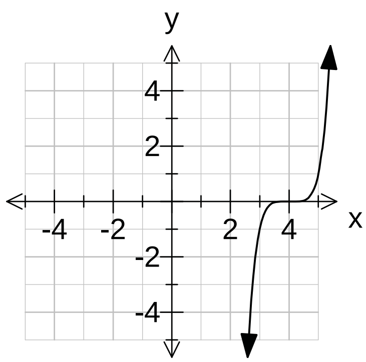
  (a) If the original dimensions of the cardboard are 30 cm by 40 cm, find the side length of the square that is cut from each corner.

  (b) Calculate the surface area.

  (ii) The actual and projected number, C (in millions), of computers sold for a region between 2010 and 2020 can be modelled by $C = 0.0092(t^3 + 8t^2 + 40t + 400)$ where $t = 0$ represents 2010. During which year are 8.51 million computers projected to be sold?

(RF11.7)

- Ask students to model the polynomial function represented by the graph as both a cubic function and a quintic function.



(RF11.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

3.4 Equations and Graphs of Polynomial Functions

SB: pp. 136-152

TR: pp. 75-79

Function Transformations

Suggested Time: 11 Hours

Unit Overview

Focus and Context

The concept of functions is one of the most important mathematical ideas students will study. Functions are used in essentially every branch of mathematics because they are an efficient and powerful way to organize and manage a variety of mathematical concepts and relationships.

In this unit, students explore the effects of horizontal and vertical translations and stretches, and reflections in the *x*-axis, the *y*-axis, and the line $y = x$ on the graphs of general functions and their related equations. In later units, they will apply these transformations to specific functions, including radical, sinusoidal, exponential, and logarithmic functions.

Students are also introduced to inverses. They identify inverse relations and inverse functions and verify that two functions are inverses of each other.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF1

Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

SCO RF2

Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

SCO RF4

Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:

- *x*-axis
- *y*-axis
- line $y = x$.

SCO RF3

Apply translations and stretches to the graphs and equations of functions.

SCO RF5

Demonstrate an understanding of inverses of relations.

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions		
RF2. Demonstrate an understanding of relations and functions. [C, R, V]	RF3. Analyze quadratics of the form $y = a(x - p)^2 + q$ and determine the: • vertex • domain and range • direction of opening • axis of symmetry • *x*- and *y*-intercepts. [CN, R, T, V] RF4. Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: • vertex • domain and range • direction of opening • axis of symmetry • *x*- and *y*-intercepts and to solve problems. [CN, PS, R, T, V] RF11. Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]	RF1. Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V] RF4. Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the: • *x*-axis • *y*-axis • line $y = x$. [C, CN, R, V] RF2. Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V] RF3. Apply translations and stretches to the graphs and equations of functions. [C, CN, R, V] RF5. Demonstrate an understanding of inverses of relations. [C, CN, R, V]

Relations and Functions

Outcomes

Students will be expected to

RF1 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

[C, CN, R, V]

Achievement Indicators:

RF1.1 *Compare the graphs of a set of functions of the form $y - k = f(x)$ to the graph of $y = f(x)$ and generalize, using inductive reasoning, a rule about the effect of k.*

RF1.2 *Compare the graphs of a set of functions of the form $y = f(x - \mathrm{h})$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of* h.

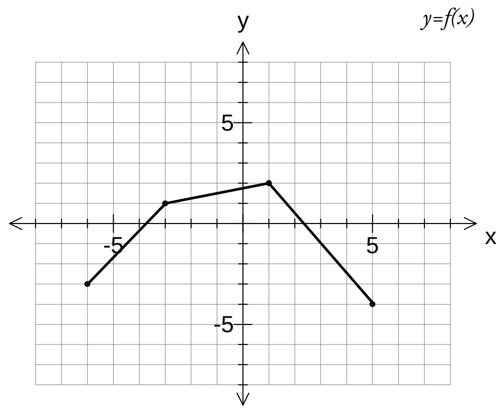
RF1.3 *Compare the graphs of a set of functions of the form $y - k = f(x - \mathrm{h})$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of* h *and k.*

RF1.4 *Sketch the graph of $y-k=f(x)$, $y=f(x-h)$ or $y-k=f(x-h)$ for given values of* h *and k, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students were introduced to quadratic functions in vertex form $y = a(x - \mathrm{h})^2 + k$ (RF3). They discovered that h and k translated the graph horizontally and vertically. These transformations allow students to identify the vertex directly from the quadratic equation. Students will now look at these translations for general functions and compare how the graph and table of values of $y = f(x)$ compares to $y - k = f(x - \mathrm{h})$.

Students should investigate the effect of changing the value of k by comparing functions of the form $y - k = f(x)$ or $y = f(x) + k$ to the graph of $y = f(x)$. Provide students with a base graph, such as the graph of $y = f(x)$ below:



Ask students to identify the key points of $y = f(x)$ and create a table of values. They should then create new tables of values and respective graphs for functions such as $y + 3 = f(x)$, $y = f(x) + 2$, and $y - 1 = f(x)$, which have a vertical translation, k. Discuss with students how this vertical translation affects the position but not the shape nor the orientation of the graph. Students should then use mapping notation $(x, y) \rightarrow (x, y + k)$ to describe the applicable vertical translation for each graph.

In a similar fashion, students should then investigate the effect of the parameter h on a function in the form $y = f(x - \mathrm{h})$, and apply the corresponding mapping notation $(x, y) \rightarrow (x + \mathrm{h}, y)$.

Students are also expected to work with functions that have been translated both horizontally and vertically when compared to a base function and to generalize those transformations using the mapping rule $(x, y) \rightarrow (x + \mathrm{h}, y + k)$. They should be exposed to both forms of the transformed function, $y - k = f(x - \mathrm{h})$ and $y = f(x - \mathrm{h}) + k$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students to describe the similarities and differences among the graphs of the following functions if they are transformations of the base function $y = f(x)$:

  (i) $y - 10 = f(x)$
  (ii) $y = f(x + 7)$

(RF1.1, RF1.2)

- Ask students to describe the translations of each function when compared to $y = f(x)$:

  (i) $y = f(x - 2)$
  (ii) $y = f(x) - 9$
  (iii) $y = f(x + 3) - 7$
  (iv) $y - 12 = f(x + 4)$

(RF1.1, RF1.2, RF1.3)

Performance

- Provide each student with a piece of graph paper and a sticky note. On the graph paper, ask students to construct a function consisting of at least four points. On the sticky note, they should write an equation in the form $y – k = f(x)$ or $y = f(x) + k$. Students should trade sticky notes and apply the transformation on the new sticky note to their own graphs.

(RF1.1)

Journal

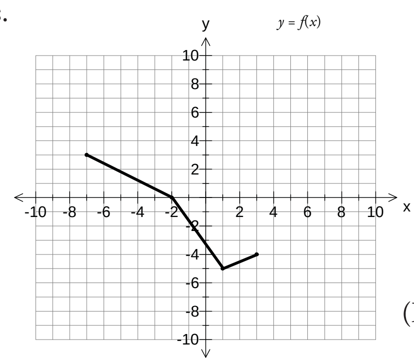
- A friend phones for help with her homework. She has a function with four key points and wants to graph $y = f(x + 5) - 7$. Ask students to describe two ways that she can create this graph.

(RF1.3, RF1.4)

Paper and Pencil

- Given the graph of $y = f(x)$, ask students to create a mapping rule and a table of values for each of the transformations below and graph the transformed functions.

  (i) $y + 2 = f(x - 6)$
  (ii) $y = f(x + 2) + 5$
  (iii) $y = f(x - 4) - 7$



(RF1.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.1 Horizontal and Vertical Translations

Student Book (SB): pp. 6-15

Teacher's Resource (TR): pp. 8-12

Relations and Functions

Outcomes

Students will be expected to

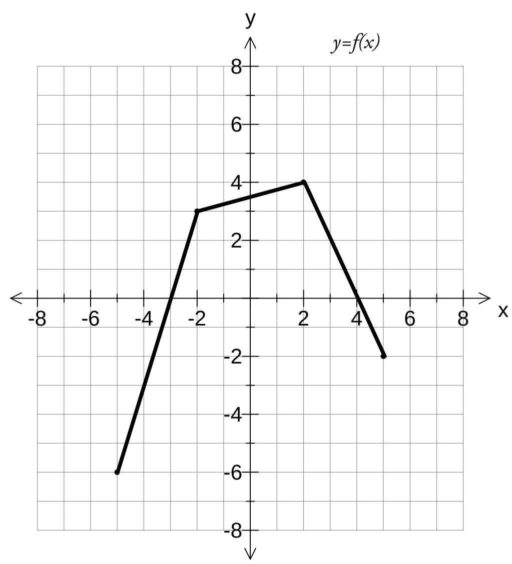
RF1 Continued ...

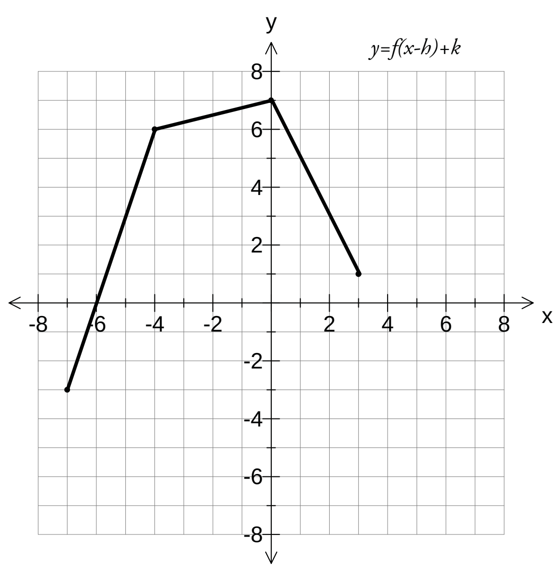
Achievement Indicator:

RF1.5 *Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y = f(x)$.*

Elaborations—Strategies for Learning and Teaching

Given the graph of a base function $y = f(x)$ and the graph of $y - k = f(x - h)$, students should identify the horizontal and vertical translations, and write the equation of the translated function. Students could be given the two graphs below and asked to describe the horizontal and vertical translations. They should then write the equation for the translated function as $y - 3 = f(x + 2)$ or $y = f(x + 2) + 3$.





General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

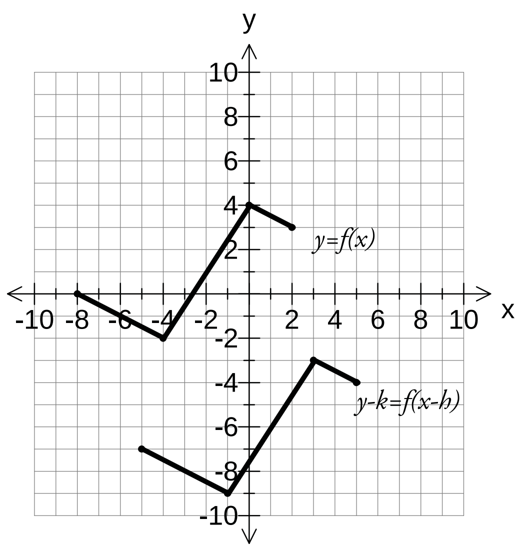
- Students could work in small groups for this activity. Provide each group with a base function $y = f(x)$ and ask each student to create their own translated graph of $y = f(x)$. They should then trade their graphs within their group and determine the equation of the function for each graph.

(RF1.5)

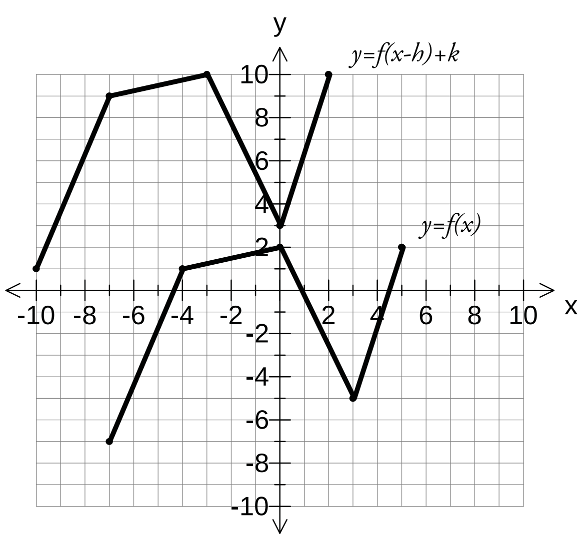
Paper and Pencil

- Ask students to determine the values of h and k and write the equation for the translated graph for each of the following:

(i)



(ii)



(RF1.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.1 Horizontal and Vertical Translations

SB: pp. 6-15

TR: pp. 8-12

Relations and Functions

Outcomes

Students will be expected to

RF4 Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:

- ***x*-axis**
- ***y*-axis**
- **line $y = x$.**

[C, CN, R, V]

Achievement Indicators:

RF4.1 *Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corrseponding ordered pair that results from a reflection through the x-axis or the y-axis.*

RF4.2 *Sketch the reflection of the graph of a function $y = f(x)$ through the x-axis or the y-axis, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.*

RF4.3 *Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ through the x-axis or the y-axis.*

RF4.4 *Sketch the graphs of the functions $y = -f(x)$ and $y = f(-x)$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.*

RF4.5 *Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$ through the x-axis or the y-axis.*

Elaborations—Strategies for Learning and Teaching

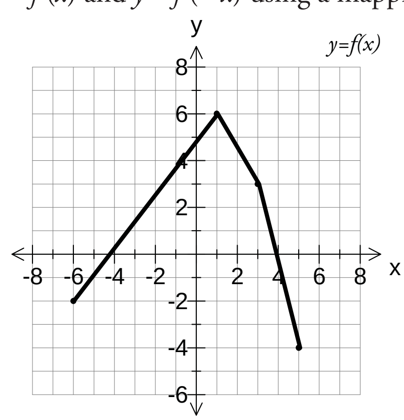
In Mathematics 2200, students were introduced to parabolas that were reflected through the *x*-axis (RF3). They will now progress to reflecting functions in both the *x*-axis and *y*-axis. The focus of this outcome is not to combine reflections with translations, but to work only with the horizontal and vertical reflections. Reflecting a point through the line $y = x$ will be addressed when students study inverses.

Students should explore the effects on the coordinates of a point when it is reflected through the *x*-axis or the *y*-axis. Ask them to plot a variety of points. As they reflect the points in the *x*-axis, they should see that the mapping rule $(x, y) \rightarrow (x, -y)$ applies. Similarly, they should conclude that the reflection of points in the *y*-axis results in the mapping rule $(x, y) \rightarrow (-x, y)$. Students then progress to sketching the reflection of a function by treating the graph as a series of points. Discuss with them how reflections, like translations, do not affect the shape of the graph. However, reflections may change the orientation of the graph.

It is important for students to recognize how the equation of $f(x)$ changes when reflections in the *x*-axis or *y*-axis occur.

- $y = -f(x)$ produces a reflection in the *x*-axis.
- $y = f(-x)$ produces a reflection in the *y*-axis.

Given the graph of a function $y = f(x)$, such as the one below, students should graph $y = -f(x)$ and $y = f(-x)$ using a mapping rule or transformations.



Similarly, given the graph of a function and a reflected graph, they should determine whether the equation of the reflected graph is of the form $y = f(-x)$ or $y = -f(x)$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students which axis the first point was reflected through to get the second point:
  (i) (6, 7) and (−6, 7)
  (ii) (−2, −7) and (−2, 7)
  (iii) (5, 0) and (−5, 0)

(RF4.1)

- Given the graph of a base function and a reflected graph, ask students how they can use key points to determine if the graph has been reflected in the *x*-axis or *y*-axis.

(RF4.3)

Journal

- Ask students to explain if it is possible for the coordinates to remain the same after a point has been reflected in an axis.

(RF4.1)

Performance

- Ask students to create the graph of a function $y = f(x)$ using at least four key points. They should trade their graphs with a partner and reflect the base function:
  (i) in the *x*-axis
  (ii) in the *y*-axis
  Ask students to write the equation of the new function.

(RF4.2, RF4.4, RF4.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.2 Reflections and Stretches

SB: pp. 16-31

TR: pp. 13-18

Relations and Functions

Outcomes

Students will be expected to

RF2 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

[C, CN, R, V]

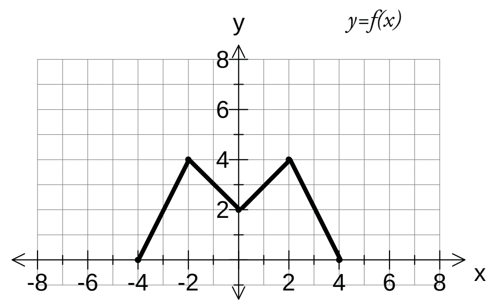
Achievement Indicator:

RF2.1 *Compare the graphs of a set of functions of the form $y = af(x)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of a.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students explored the effect of a vertical stretch on quadratic functions (RF3). They will now work with both vertical and horizontal stretches for general functions. Students should describe how these stretches change the shape of the graph and determine these stretches when given the graphs of the base function and the transformed function. This outcome focuses on horizontal and vertical stretches, so at this point, students should not encounter functions that have been both stretched and translated.

Students should recognize the effect of $|a|$ as a vertical stretch of the function $y = f(x)$ in the form $y = af(x)$ or $\frac{y}{a} = f(x)$. They could be given a graph of $f(x)$ as shown.



Using key points, ask them to generate the table of values for $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.

Students should notice that a changes the shape of the function by stretching the graph vertically. They should compare the key points of the original function to the points of the transformed function to generalize the mapping rule $(x, y) \to (x, ay)$.

When creating the table of values and graph of the transformed function, students may notice that some points did not change even after a vertical stretch is applied. Points that do not change after a transformation has been applied are called invariant points. Students were exposed to these points in Mathematics 2200 when they worked with reciprocal functions (RF11).

Students should also work with values of a that are negative. It should be noted that a negative a value will cause a reflection in the x-axis, and that the vertical stretch is positive; that is, the vertical stretch is $|a|$. Students should compare relations where $|a| > 1$ to those where $|a| < 1$ to see how these values of a influence the graph.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

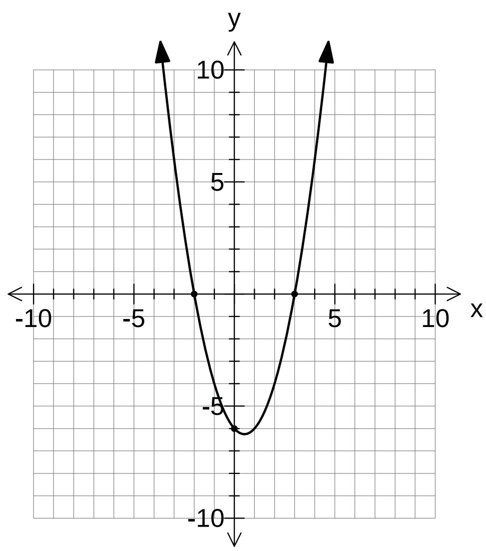
Interview

- Ask students to explain why the vertical stretch factor for the function $y = af(x)$ is determined using $|a|$. The following questions could also be posed to lead to further elaboration: "Is it possible to have a negative vertical stretch factor? If stretches are always positive, what does a negative value for a indicate about the graph?"

(RF2.1, RF4.1, RF4.2, RF4.3)

Journal

- Given the graph of the function below, ask students to explain which points are invariant points and why they do not change after the application of a vertical stretch.



(RF2.1)

- Ask students to explain what the effects will be on the graph of a function when $|a| > 1$ or $|a| < 1$.

(RF2.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.2 Reflections and Stretches

SB: pp. 16-31

TR: pp. 13-18

Relations and Functions

Outcomes

Students will be expected to

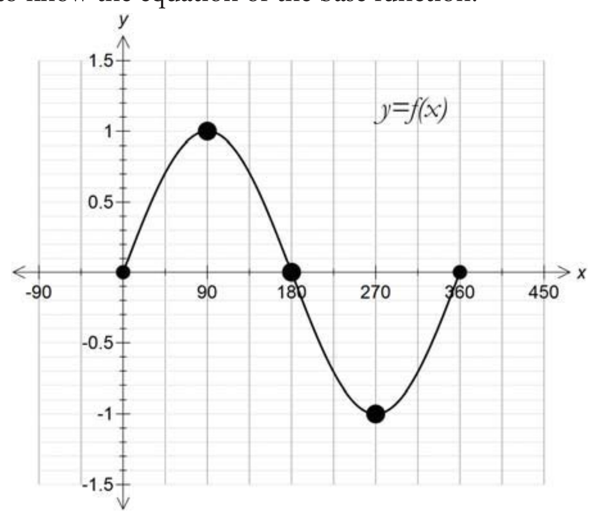
RF2 Continued ...

Achievement Indicator:

RF2.2 *Compare the graphs of a set of functions of the form $y = f(bx)$ to the graph of $y = f(x)$ and generalize, using inductive reasoning, a rule about the effect of b.*

Elaborations—Strategies for Learning and Teaching

This will be students' first exposure to the concept of horizontal stretch. Graphing technology could be used to display a base function such as $y = \sin(x)$ with a restricted domain of $[0°, 360°]$. It is not necessary for students to know the equation of the base function.



Providing students with the graphs of $y = f(3x)$ and $y = f\left(\frac{1}{2}x\right)$, they can examine the effect on the graph and the table of values. Students should discuss the general effects of a horizontal stretch on the graph of the base function and the table of values. The general mapping $(x, y) \rightarrow \left(\frac{1}{b}x, y\right)$ from the base function to the transformed function can be inferred from the example. Students should compare relations where $|b| < 1$ to those for which $|b| > 1$. It should also be noted that if $b < 0$, the graph will be stretched, as well as reflected in the y-axis. As with vertical stretches, horizontal stretches are always positive.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

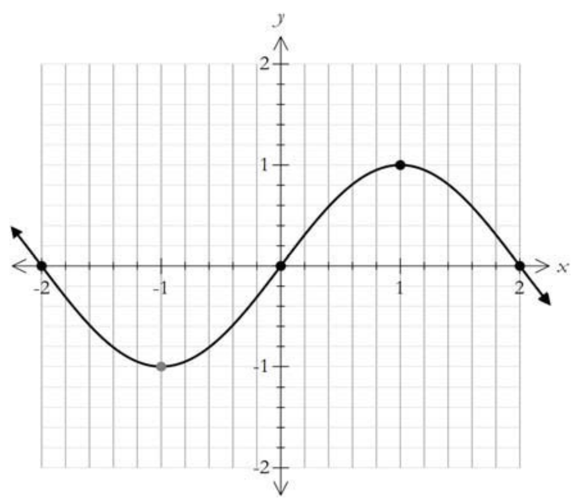
Interview

- Ask students to explain why the horizontal stretch factor is given by $\frac{1}{|b|}$ for the function $y = f(bx)$.

(RF2.2)

Journal

- Given the graph of the function below, ask students to explain which points are invariant points and why they do not change after the application of a horizontal stretch.



(RF2.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.2 Reflections and Stretches

SB: pp. 16-31

TR: pp. 13-18

Relations and Functions

Outcomes

Students will be expected to

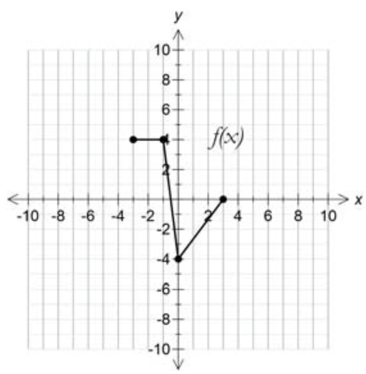
RF2 Continued ...

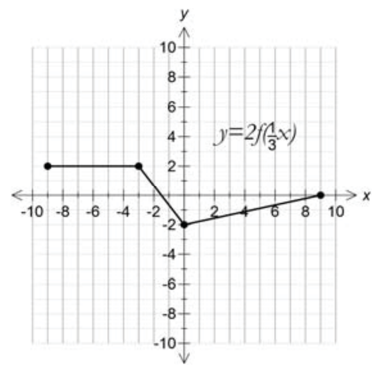
Achievement Indicator:

RF2.3 *Compare the graphs of a set of functions of the form y = a f(bx) to the graph of y = f (x), and generalize, using inductive reasoning, a rule about the effects of a and b.*

Elaborations—Strategies for Learning and Teaching

Students should also work with functions that have both a vertical and horizontal stretch. By comparing two graphs, they should be able to determine the effects that the values of a and b have on the graph of $y = af(bx)$ when compared to $y = f(x)$. Students could be given the two graphs below, for example, and asked the following questions:





- What effect did the 2 have on the graph?
- What effect did the 2 have on the table of values?
- What effect did the $\frac{1}{3}$ have on the graph?
- What effect did the $\frac{1}{3}$ have on the table of values?

Students should then generalize a mapping rule for functions that have a vertical stretch of $|a|$ and a horizontal stretch of $\frac{1}{|b|}$ as $(x, y) \rightarrow \left(\frac{x}{b}, ay\right)$. It should be noted that the focus on vertical and horizontal stretch should be with functions that are bounded.

Students could be introduced to unbounded functions such as $y = x^2$ and $y = |x|$ and explore transformations of these functions with both the vertical and horizontal stretch. For many of the unbounded functions that students have already seen, stretches can be described as either a horizontal stretch or a vertical stretch. Using a function such as $y = 4x^2$, which has a vertical stretch of 4, students can compare the effect of a vertical stretch and a horizontal stretch. If this function is written as $y = (2x)^2$, there is a horizontal stretch of $\frac{1}{2}$. Encourage students to create a mapping rule for both functions and produce a table of values and graph from the base function $y = x^2$ to verify that these stretches produce the same effect.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Using technology, ask students to demonstrate the effects of stretching an image for different values of *a* and *b* in various combinations.

(RF2.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.2 Reflections and Stretches

SB: pp. 16-31

TR: pp. 13-18

Relations and Functions

Outcomes

Students will be expected to

RF2 Continued ...

Achievement Indicators:

RF2.4 *Sketch the graph of $y = af(x)$, $y = f(bx)$ or $y = af(bx)$ for given values of a and b, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.*

RF2.5 *Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.*

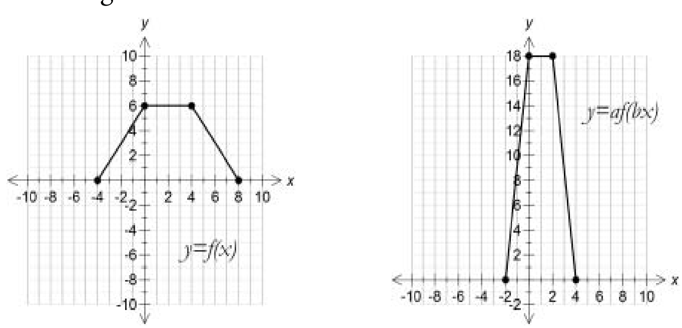
Elaborations—Strategies for Learning and Teaching

Given the graph of a function, students should apply a horizontal stretch and/or a vertical stretch to produce the graph of the transformed function $y = af(bx)$. Two possible ways to produce the graph are:

- to identify key points and use a mapping rule to create a new table of values
- to use knowledge of stretches to transform the graph directly.

Given the graph of a transformed function and a base function to compare it with, students should determine the horizontal and vertical stretch, and then write the equation of the transformed function in terms of $y = af(bx)$ or $\frac{y}{a} = f(bx)$. The focus of this outcome is not to determine the specific equation for the base function or transformed function such as $\frac{y}{7} = |4x|$, but to work with general equations like $\frac{y}{7} = f(4x)$.

The stretches can be determined by comparing the domain and range of the functions. Students should be exposed to graphs such as the following:



The domain of $y = f(x)$ is $[-4, 8]$, which has a span of 12 units. The domain of $y = af(bx)$ is $[-2, 4]$, which has a span of 6 units. Students should realize that the horizontal stretch is $\frac{1}{2}$, which means $|b| = 2$.

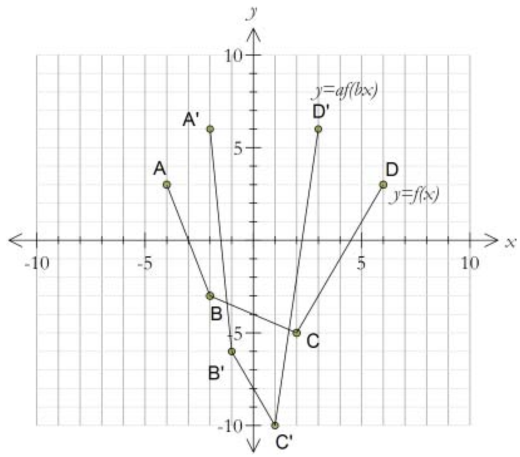
Similarly, the range of $y = f(x)$ is $[0, 6]$, a span of 6 units, and the range of $y = af(bx)$ is $[0, 18]$, a span of 18 units. Students should realize that the vertical stretch is 3, which means $|a| = 3$. Since there are no reflections, they can conclude that the equation of the transformed graph is $y = 3f(2x)$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to write the general equation, in terms of $f(x)$ for the stretched image shown on the same set of axes, by determining the values of a and b from the graph.



(RF2.5)

Journal

- Ask students to explain when the x-intercepts are invariant for a stretch, and when the y-intercepts are invariant for a stretch. They could also be asked to describe the circumstances under which a point would be invariant for both types of stretches.

(RF2.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.2 Reflections and Stretches

SB: pp. 16-31

TR: pp. 13-18

Relations and Functions

Outcomes

Students will be expected to

RF3 Apply translations and stretches to the graphs and equations of functions.

[C, CN, R, V]

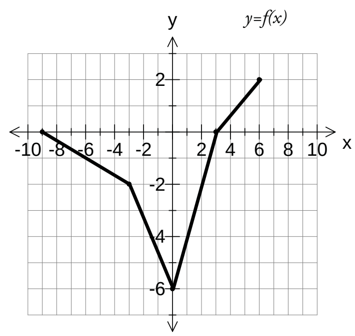
Achievement Indicator:

RF3.1 *Sketch the graph of the function $y - k = af(b(x - h))$ for given values of a, b, h and k, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.*

Elaborations—Strategies for Learning and Teaching

Students have worked with translations, reflections in the *x*- and *y*-axis, and stretches. For the most part, these transformations have been addressed independently of one another. They will now extend their work to functions and graphs that have all types of transformations.

Given the graph of the function $y = f(x)$, students should graph the function $y - k = af(b(x - h))$ for different values of *a*, *b*, h, and *k*. The graph could be created using transformations. Using the graph of a function, such as $y = f(x)$ shown here, ask students to graph a transformed function such as $y = -2f\left(3\left(x-1\right)\right)+4$.



Remind them of the importance of the order of operations. Since stretches and reflections are the result of multiplication and translations are the result of addition, the stretches and reflections are applied first. They apply the following transformations to each point to produce the transformed graph:

- horizontal stretch of $\frac{1}{3}$
- vertical stretch of 2
- reflection in the *x*-axis
- horizontal translation of 1 unit right
- vertical translation of 4 units up

Students should note that the stretches and reflections can be applied in any order, as long as it is before the translations. Similarly, the order in which the translations are applied is not important, as long as they are applied after the stretches and reflections.

Students could also generate a table of values based on the key points, and use the mapping $\left(x,y\right)\rightarrow\left(\frac{1}{3}x+1,-2y+4\right)$ to generate a new table of values.

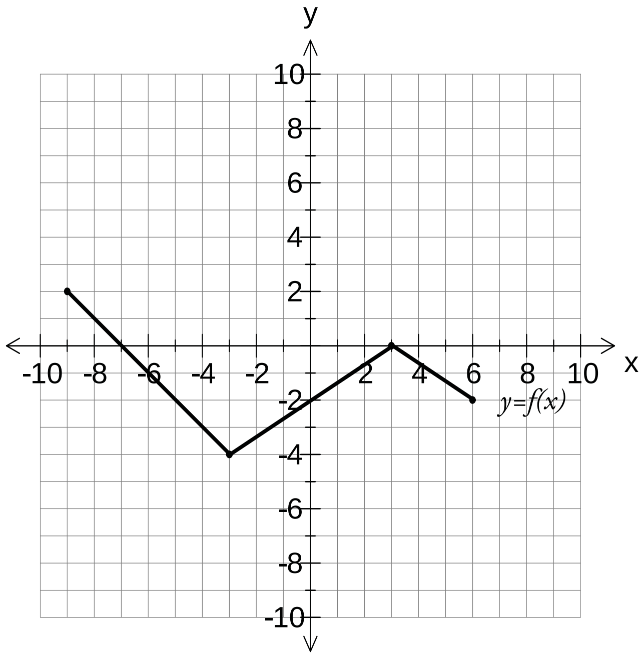
It is sometimes necessary to rewrite a function before it can be graphed. Before graphing $y - 6 = 3f(4x - 8)$, for example, students should write the function as $y - 6 = 3f(4(x - 2))$. This will help them correctly identify the value of h as 2, rather than 8.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Given the graph of the function $y = f(x)$ shown, ask students to sketch the graph of $y = 2f(-3(x+1)) - 2$.



(RF3.1)

Journal

- Ask students to describe the steps in graphing a function of the form $y = af(b(x - h)) + k$, where the shape of the graph of $f(x)$ is well-known.

(RF3.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.3 Combining Transformations

SB: pp. 32-43

TR: pp. 19-24

Relations and Functions

Outcomes

Students will be expected to

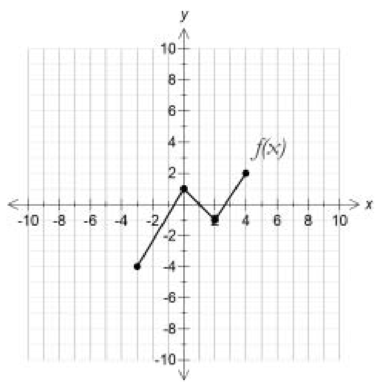
RF3 Continued ...

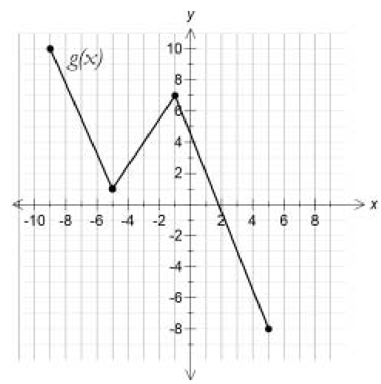
Achievement Indicator:

RF3.2 *Write the equation of a function, given its graph which is a translation and/or stretch of the graph of the function $y = f(x)$.*

Elaborations—Strategies for Learning and Teaching

Students should compare the graph of a base function with the graph of a transformed function, identify all transformations, and state the equation of the transformed function. The focus should be on functions that have a bounded domain and range. In an example such as the following students should find the equation of $g(x)$ as a transformation of $f(x)$:





The following prompts could be used to initiate discussion:

- What is the domain of each function? How will this help in identifying the horizontal stretch?

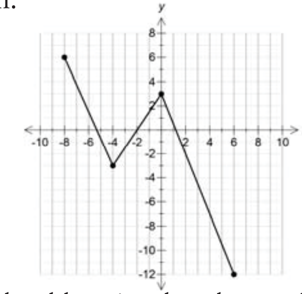
  Students should recognize that the domain of $f(x)$ is $[-3, 4]$ which has a span of 7 units, and that the domain of $g(x)$ is $[-9, 5]$ which has a span of 14 units. Therefore, $g(x)$ has a horizontal stretch of 2.

- What is the range of each function? How will this help in identifying the vertical stretch?

  By comparing the ranges in a similar fashion, students should recognize the vertical stretch is 3.

- Is the graph reflected in the y-axis? Explain your reasoning.

Student findings should result in $a = 3$ and $b = -\frac{1}{2}$. Applying these transformations to $f(x)$ produces the graph:



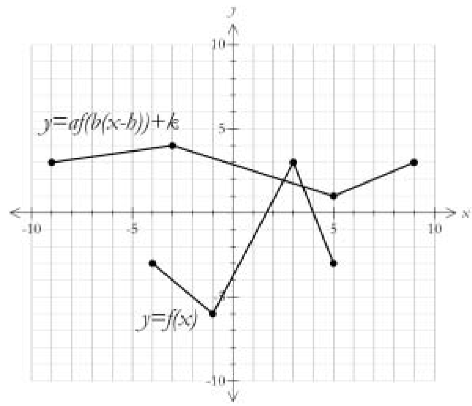
As students compare the key points, they should notice that the graph must be shifted 1 unit left and 4 units up to produce $g(x)$, resulting in the function $y = 3f\left(-\frac{1}{2}(x+1)\right) + 4$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Using a graph such as the one shown, ask students to determine the specific equation for the image of $y = f(x)$ in the form $y = af(b(x-h)) + k$.



(RF3.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.3 Combining Transformations

SB: pp. 32-43

TR: pp. 19-24

Relations and Functions

Outcomes

Students will be expected to

RF5 Demonstrate an understanding of inverses of relations.

[C, CN, R, V]

RF4 Continued ...

Achievement Indicators:

RF4.6 *Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection through the line y = x.*

RF5.1 *Explain how the transformation (x, y) → (y, x) can be used to sketch the inverse of a relation.*

RF5.2 *Explain the relationship between the domains and ranges of a relation and its inverse.*

Elaborations—Strategies for Learning and Teaching

As students begin work with inverse relations, they will explore the relationship between the graph of a relation and its inverse, and determine whether a relation and its inverse are functions. Students produce the graph of an inverse from the graph of the original relation, restrict the domain of a function so that its inverse is also a function, and determine the equation for f^{-1} given the equation for *f*.

Discuss with students how an inverse of a relation 'undoes' whatever the original relation did. Using examples from non-mathematical situations may provide a better understanding of a topic that is otherwise very abstract. The inverse of opening a door is closing the door; the inverse of wrapping a gift is unwrapping a gift. From a mathematics perspective, encourage students to think of inverse relations as undoing all of the mathematical operations. The following table helps to illustrate this concept:

Function	Inverse
$f(x)=x+2$	$f^{-1}(x)=x-2$
$f(x)=5x$	$f^{-1}(x)=\frac{x}{5}$
$f(x)=x^2$	$f^{-1}(x)=\sqrt{x}$

Students should compare a function to its inverse using a table of values. Examining a table of values for $y = x + 4$ and its inverse, $y = x - 4$, shows that the *x* and *y* values are interchanged.

x	y
–2	2
–1	3
0	4
1	5
2	6

x	y
2	–2
3	–1
4	0
5	1
6	2

From this, students should see the mapping notation $(x, y) \rightarrow (y, x)$ as a reversal of the *x* and *y* values in order to represent an undoing of a process. The input of the function is the output of the inverse, and vice versa. This leads to the relationship between the domains and ranges of a relation and its inverse. Students may have difficulty generalizing this relationship for linear functions, as the domain and range are all real numbers. Use the graph of a quadratic function, such as $y=(x-2)^2+1$, and the graph of its inverse to highlight the relationship. The original function has all real numbers in its domain and the range is $\{y \mid y \geq 1, y \in R\}$. They should note that the domain and range are interchanged for the inverse.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

- Students respond to three reflective prompts that describe what they learned about inverse relations.

  Example of a 3-2-1 reflection sheet:

3 new things I learned 1. 2. 3. **2** things I am still struggling with 1. 2. **1** thing that will help me tomorrow 1.

  Provide students with a copy of the reflection sheet and time to complete their reflections. They could also be paired up to share their 3-2-1 reflections.

  (RF4.6, RF5.1, RF5.2, RF5.3)

- Ask students to explain what it means in theory for one function to be the inverse of another in terms of their domains and ranges.

  (RF5.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.4 Inverse of a Relation

SB: pp. 44-55

TR: pp. 25-30

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

RF5 Continued ...

Achievement Indicators:

RF5.3 *Explain how the graph of the line y = x can be used to sketch the inverse of a relation.*

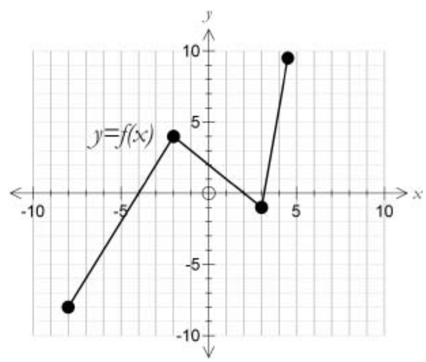
RF5.4 *Sketch the graph of the inverse relation, given the graph of a relation.*

RF4.7 *Sketch the reflection of the graph of a function y = f (x) through the line y = x, given the graph of the function y = f (x), where the equation of y = f (x) is not given.*

RF4.8 *Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function y = f (x) through the line y = x.*

Elaborations—Strategies for Learning and Teaching

By having students use $(x, y) \rightarrow (y, x)$ to create a table of values for the inverse of a function, they should quickly realize that there is reflective symmetry when the graphs of f and f^{-1} are sketched on the same set of axes. The axis of symmetry becomes apparent when they draw the line $y = x$. As an example, a graph such as the following could be presented:



Based on the graph, ask students to complete the following:

- Create a table of values for the function using the ordered pairs for the four key points shown.
- Transform the table by applying the mapping $(x, y) \rightarrow (y, x)$.
- Graph the relation represented by the new table.
- Look for a relationship between the graphs of f and f^{-1}.
- Generalize the pattern to a relationship between graphs of functions and their inverses in general.

After seeing the graphical relationship between a relation and its inverse, the goal is for students to take a graph of a relation and produce a graph of its inverse on the same set of axes without having to go through the process of producing tables of values.

It should be noted that a common error occurs in the notation used for inverse functions. Students may incorrectly write $f^{-1}(x)$ as $\frac{1}{f(x)}$.

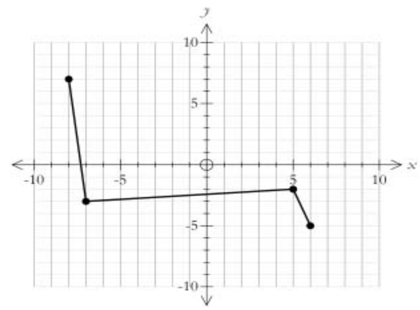
General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

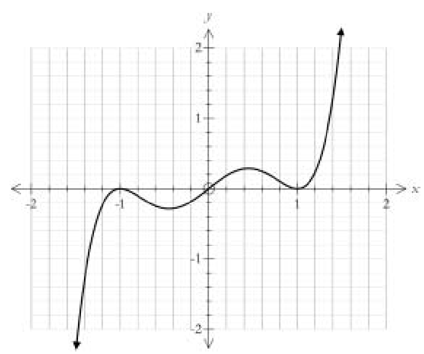
Paper and Pencil

- For each graph shown, ask students to sketch the inverse relation on the same set of axes.

(i)



(ii)

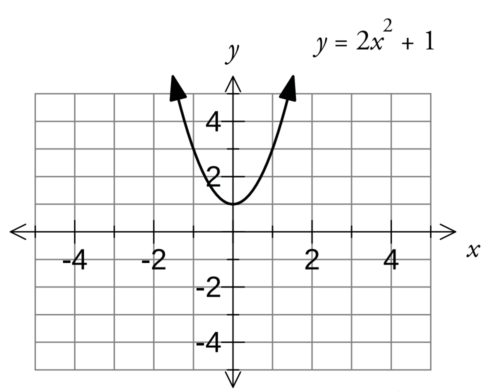


(RF5.4)

Journal

- Students should discuss how they could sketch the graph of the inverse of the relation graphed below, using both indicated approaches:

  (i) the line $y = x$
  (ii) the mapping rule $(x, y) \rightarrow (y, x)$



(RF5.1, RF5.3, RF4.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.4 Inverse of a Relation

SB: pp. 44-55

TR: pp. 25-30

Relations and Functions

Outcomes

Students will be expected to

RF5 Continued ...

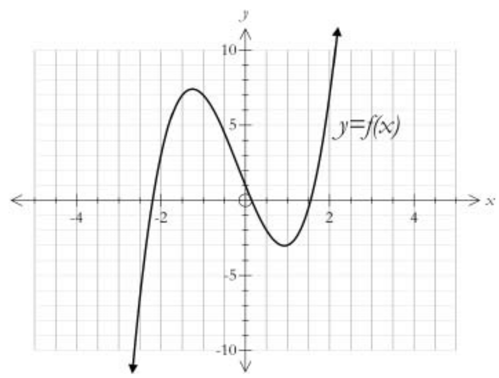
Achievement Indicator:

RF5.5 *Determine if a relation and its inverse are functions.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students differentiated between a function and a relation (RF2). They also used the vertical line test to determine if a relation is also a function. This content could be quickly reviewed to begin a discussion on determining if the inverse of a function is also a function.

Students could be given a graph such as the following, and asked to complete the tasks below.



- Use a vertical line test to determine if $y = f(x)$ is a function.
- Construct the graph of $y = f^{-1}(x)$ by reflecting the graph of $y = f(x)$ in the line $y = x$.
- Use the vertical line test to determine if $y = f^{-1}(x)$ is a function.
- What kind of line test could be used with the graph of $y = f(x)$ to determine if its inverse would be a function?

Students should conclude that the horizontal line test is used to determine if an inverse relation is a function.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

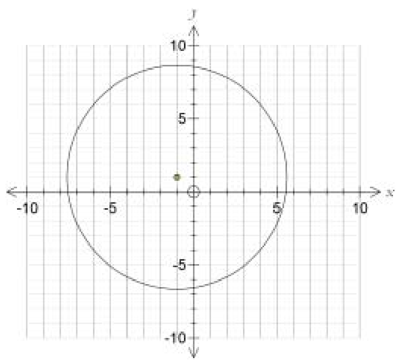
- Ask students to explain the difference between a vertical line test and a horizontal line test in terms of what each one is used to determine, and why they work.

(RF5.5)

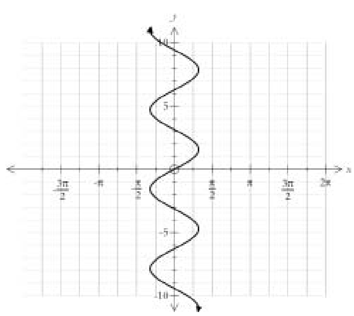
Paper and Pencil

- Ask students to determine whether the inverse of each relation graphed here is a function, without actually sketching it.

(i)



(ii)



(RF5.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.4 Inverse of a Relation

SB: pp. 44-55

TR: pp. 25-30

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

RF5 Continued ...

Achievement Indicators:

RF5.6 *Determine restrictions on the domain of a function in order for its inverse to be a function.*

RF5.7 *Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.*

RF4.9 *Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$ through the line $y = x$.*

RF5.8 *Determine, algebraically or graphically, if two functions are inverses of each other.*

Elaborations—Strategies for Learning and Teaching

Students are required to restrict the domain of functions to ensure that the inverse is also a function. Examples should be limited to linear and quadratic functions. Students should realize that the only linear functions whose inverses are not functions are those that have graphs which are horizontal lines.

In Mathematics 2200, students converted the equations of quadratic functions from standard form, $y = ax^2 + bx + c$, to vertex form, $y = a(x - h)^2 + k$, by completing the square (RF4) and graphed quadratic functions in vertex form (RF3). Students continue to convert forms of quadratic equations, in order to identify the vertex of the function so that its domain may be restricted. Students could be guided through the following process for the quadratic function represented by $f(x) = 2x^2 - 8x + 11$:

- write the equation in vertex form, and sketch its graph.
- use the graph of the quadratic to sketch its inverse on the same set of axes.
- determine if the inverse relation is also a function. If not, consider what would need to change about the graph of the original quadratic in order for its inverse to be a function.
- restrict the domain of the function $y = f(x)$ so that $y = f^{-1}(x)$ is also a function.

Restricting the domain of a parabola so that $x > h$ produces a function whose inverse is also a function. Restricting the domain so that $x < h$ is also acceptable, although not as common.

Students should then be shown the process by which interchanging the x and y variables in the equation and then rearranging to isolate y produces the equation for $y = f^{-1}(x)$.

When presented with two functions, students should also determine whether or not they are inverses of one another. This can be done on a graph that displays both functions by sketching the line $y = x$ and deciding if the functions are mirror images of one another.

Algebraically, students should be given the equation representing each function. They determine the equation of the inverse of one of the given functions, and then decide if it is equivalent to the other given function.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the equation of the inverse for each of the following relations:
  (i) $y = 5x$
  (ii) $y = 3x - 4$
  (iii) $y = -2x + 5$

  (RF5.7)
- Ask students to state the restricted domain for each of the following relations so that the inverse relation is a function, and write the equation of the inverse:
  (i) $y = x^2 - 6x + 10$
  (ii) $y = 5x^2 + 20x - 9$
  (iii) $y = 2x^2 - 8x + 1$

  (RF5.6, RF5.7)
- Ask students to determine if the inverses of the following functions are functions. If the inverse is not a function, they should describe how they can be modified to become functions.
  (i) $f(x) = 3x^2$
  (ii) $f(x) = x^2 + 2x$
  (iii) $f(x) = 2x^2 + 4$

  (RF5.5, RF5.6)
- Students match each of the equations from the first list with its inverse in the second list:

Function	Inverse
$y = 4x - 1$	$y = \frac{x-16}{4}$
$y = x^2 + 8x + 2,\ x \geq -4$	$y = \frac{x+1}{4}$
$y = 3x^2 - 12x + 15,\ x \geq 2$	$y = \sqrt{x+14} - 4$
	$y = \sqrt{\frac{x-3}{3}} + 2$

(RF5.8)

Performance

- Create square sheets of paper with graphs of pairs of functions. Create the pairs in such a way that some are inverses of one another while others are not. Construct the grid on each graph so that the scales are identical on each axis, and so that the x- and y-axes have the same limits. Distribute one square to each student and ask them to fold their papers in a way that would determine whether or not the graphs are inverses of one another. Ask them to explain their method. (RF5.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

1.4 Inverse of a Relation

SB: pp. 44-55

TR: pp. 25-30

Note:

The examples in the student book on pp. 49-50 are limited to quadratics of the form $y = ax^2 + c$.

Students should also be exposed to equations of the form $y = ax^2 + bx + c$, as on p. 53 - #10 and p. 54 - #12.

Radical Functions

Suggested Time: 8 Hours

Unit Overview

Focus and Context

Work with function transformations continues in this unit with a specific focus on radical functions. Students sketch graphs of radical functions by applying translations, stretches and reflections to the graph of $y = \sqrt{x}$. They analyze transformations to identify the domain and range of radical functions.

In this unit, the graph of $y = \sqrt{f(x)}$ is related to the graph of $y = f(x)$ and the domain and range for each function are compared. Finding approximate solutions to radical equations is also addressed graphically.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF12

Graph and analyze radical functions (limited to functions involving one radical).

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions		
AN2. Demonstrate an understanding of irrational numbers by: • representing, identifying and simplifying irrational numbers • ordering irrational numbers. [CN, ME, R, V] RF6. Relate linear relations expressed in: • slope-intercept form $y = mx + b$ • general form Ax + By + C = 0 • slope-point form $y - y_1 = m(x - x_1)$ to their graphs. [CN, R, T, V]	AN2. Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. [CN, ME, PS, R] AN3. Solve problems that involve radical equations (limited to square roots). [C, PS, R] RF3. Analyze quadratics of the form $y = a(x - p)^2 + q$ and determine the: • vertex • domain and range • direction of opening • axis of symmetry • x- and y-intercepts and to solve problems. [CN, R, T, V] RF5. Solve problems that involve quadratic equations. [C, CN, PS, R, T, V]	RF12. Graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]

Relations and Functions

Outcomes

Students will be expected to

RF12 Graph and analyze radical functions (limited to functions involving one radical).

[CN, R, T, V]

Achievement Indicators:

RF12.1 *Sketch the graph of the function* $y=\sqrt{x}$*, using a table of values, and state the domain and range.*

RF12.2 *Sketch the graph of the function* $y-k=a\sqrt{b(x-\mathrm{h})}$ *by applying transformations to the graph of the function* $y=\sqrt{x}$ *, and state the domain and range.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students worked with radicals and radical expressions with numerical and variable radicands (AN2), and solved equations and problems dealing with radical equations (AN3). Students now use their knowledge of transformations from the previous unit (RF4) to graph radical functions. They also determine the domain and range of radical functions.

While students have been introduced to radical equations (AN3), they may or may not have been exposed to what the graphs of radical functions look like. They should be introduced to the graphs by creating the graph of $y=\sqrt{x}$ using a table of values. Once the graph is produced, the domain and range of $y=\sqrt{x}$ should be discussed. Students should note that because it is not possible to take the square root of negative numbers in the real number system, the radicand must be greater than or equal to zero, resulting in a domain of $x \in [0, \infty)$. Technology could also be used to create the table of values and the graph.

Once students are familiar with the characteristics of the graph of $y=\sqrt{x}$, the characteristics of a transformed radical function of the form $y-k=a\sqrt{b(x-\mathrm{h})}$ or $y=a\sqrt{b(x-\mathrm{h})}+k$ should be examined. The graphs can be produced by:

- applying the transformations
- creating a mapping rule, deriving a table of values, and plotting the points.

Students have already been exposed to these graphing techniques using general functions (RF4). They should now apply these techniques to the specific function $y=\sqrt{x}$.

Students could explore a radical function such as $y-1=-3\sqrt{\frac{1}{2}(x+6)}$. Using transformations, they should see that:

- $a = -3$, so all points have a vertical stretch of 3
- Since $a < 0$, all points are reflected in the x-axis
- $b = 2$, so all points have a horizontal stretch of 2
- h $= -6$, so all points have a horizontal translation of 6 units left
- $k = 1$, so all points have a vertical translation of 1 unit up.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

- Ask students to describe the similarities and differences between the graphs of $y=\sqrt{x}$ and $y=\frac{1}{3}\sqrt{-2(x+1)}-4.$ They should discuss the domain and range of each function.

(RF12.1, RF12.2)

Paper and Pencil

- Ask students to create the graph of $y-3=-\sqrt{\frac{1}{2}(x-1)}$, describing all transformations and stating the domain and range.

(RF12.2)

Performance

- Create several pairs of cards where one card contains the equation of a radical function, and the second card contains the range. Distribute the cards among the students and have them find their partner by matching the radical function with its range. Once students have found their partners, they should create the graph of the radical function and find its domain.

(RF12.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

2.1 Radical Functions and Transformations

Student Book (SB): pp. 62-77

Teacher's Resource (TR): pp. 38-43

Relations and Functions

Outcomes

Students will be expected to

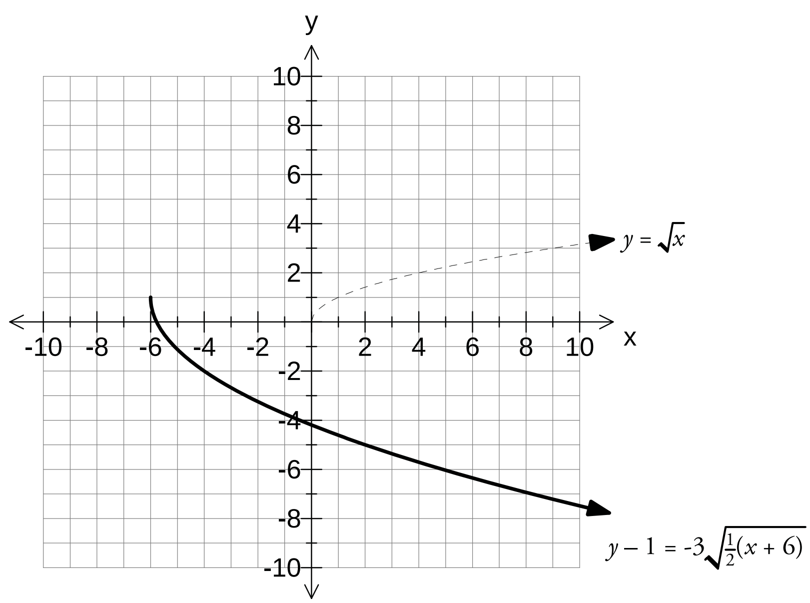
RF12 Continued ...

Achievement Indicator:

RF12.2 *Continued*

Elaborations—Strategies for Learning and Teaching

When students create the graph by applying transformations, remind them that they must apply stretches and reflections first, and translations last.



Students could also write the mapping rule for the function, transform the points of $y = \sqrt{x}$, and plot these points to create the graph.

Students should determine the domain and range of the function from the graph. After they have worked through a number of examples, they should see a pattern that allows them to determine the domain and range of a radical expression such as $y = 4\sqrt{-2(x+5)} + 7$ without creating an accurate graph. This can be done by examining the reflections and translations. They should realize that stretches do not affect the domain and range of radical functions.

It is important to note that students are not expected to determine the equation of a radical function given its graph or points on the graph.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to sketch the graphs of $f(x)=\sqrt{x-3}$ and $g(x)=-\sqrt{x-3}$ on the same coordinate plane. They should notice that the two graphs together form a parabola. Ask them to explain why a quadratic equation cannot be used to define *y* as a function of *x* for the resulting graph.

(RF12.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

2.1 Radical Functions and Transformations

SB: pp. 62-77

TR: pp. 38-43

Note

SB: p. 68 - Example 3, p. 74 - #10

Students are not expected to determine a radical function from a graph.

Relations and Functions

Outcomes

Students will be expected to

RF12 Continued ...

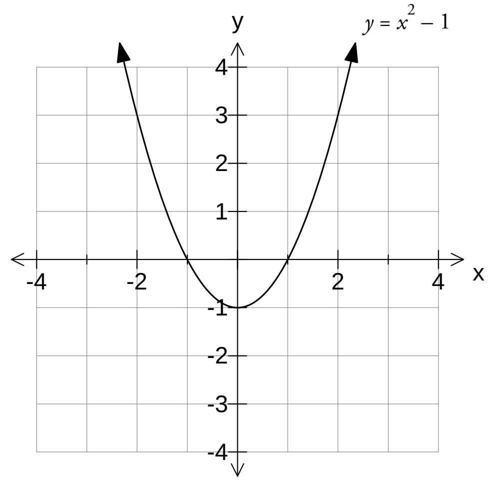
Achievement Indicators:

RF12.3 *Sketch the graph of the function $y=\sqrt{f(x)}$, given the equation or graph of the function y = f(x), and explain the strategies used.*

RF12.4 *Compare the domain and range of the function $y=\sqrt{f(x)}$, to the domain and range of the function y = f(x), and explain why the domains and ranges may differ.*

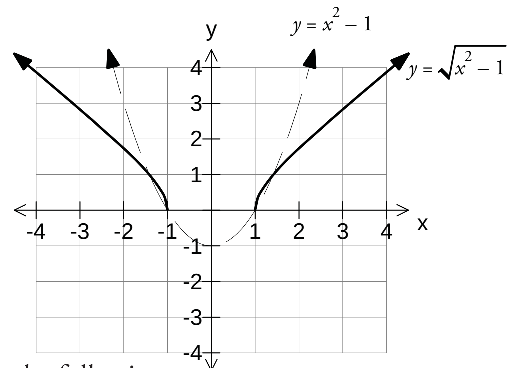
Elaborations—Strategies for Learning and Teaching

Given the graph of $y = f(x)$, students should graph $y=\sqrt{f(x)}$. Ask them to consider, for example, the graph of $y = x^2 - 1$.



One method to produce the graph of $y=\sqrt{x^2-1}$ is to first generate a table of values for $y=x^2-1$ from the given graph. Then, to graph $y=\sqrt{x^2-1}$, students take the square root of the y-values.

x	$f(x)$	$\sqrt{f(x)}$
-2	3	$\sqrt{3}$
-1	0	0
0	-1	undefined
1	0	0
2	3	$\sqrt{3}$



Ask students to think about the following:

- Why is the graph undefined from $x \in (-1, 1)$?
- Are there any invariant points? If so, what are they?
- Where is the graph of $y=\sqrt{f(x)}$ above $y = f(x)$? Below $y = f(x)$?

Students should realize that $y=\sqrt{f(x)}$ is undefined where $f(x) < 0$, that the invariant points occur where $f(x) = 0$ or $f(x) = 1$, and that $\sqrt{f(x)}$ is above the graph of $y = f(x)$ where $0 < f(x) < 1$. These results, along with other key points, can then be used to help create the graph of $y=\sqrt{f(x)}$ when $y = f(x)$ is given.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

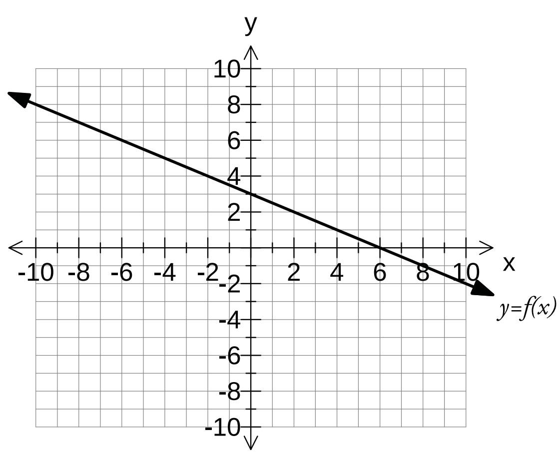
- Without graphing, ask students to determine the domain and range of each function:

  (i) $y = \sqrt{x^2 + 4x - 5}$

  (ii) $y = \sqrt{-\frac{1}{2}x^2 + x + 12}$

(RF12.4)

- Given the graph of $f(x)$ below, students should create the graph of $y = \sqrt{f(x)}$ by examining the intercepts and the invariant points:



(RF12.3)

Journal

- Ask students to explain why the domain of $y = \sqrt{x^2 + 4}$ is $\{x \mid x \in R\}$, but the domain of $y = \sqrt{x^2 - 4}$ is $\{x \mid x \leq -2 \text{ or } x \geq 2, x \in R\}$.

(RF12.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

2.2 Square Root of a Function

SB: pp. 78-89

TR: pp. 44-49

Relations and Functions

Outcomes

Students will be expected to

RF12 Continued ...

Achievement Indicators:

RF12.3, RF12.4 *Continued*

Elaborations—Strategies for Learning and Teaching

Students could also be given the equation $y = f(x)$ and use it to generate a table of values. From this, they can then graph $y=\sqrt{f(x)}$. The graph and table of values of $y = f(x)$ can also be generated with the use of technology.

The graphs of $y = f(x)$ are limited to linear and quadratic functions.

Students can use the graphs of $y = f(x)$ and $y=\sqrt{f(x)}$ to determine the domain and range of both functions. When equations are given for the functions, they can first graph each function and then determine the domain and range.

An alternative method to determine domain and range of $y = f(x)$ and $y=\sqrt{f(x)}$ involves analyzing key points. Given $y = -x^2 + 6x - 5$, for example, students can use knowledge of quadratic functions from Mathematics 2200 to determine the x- and y-intercepts of the parabola (RF5), and the vertex of the parabola (RF3). This information can then be used to identify key points on the graph of $y=\sqrt{-x^2+6x-5}$:

Function	$y=-x^2+6x-5$	$y=\sqrt{-x^2+6x-5}$
x-intercepts	1 and 5	1 and 5
y-intercepts	-5	none
Maximum value	$(3,4)$	$(3,2)$
Minimum value	none	0

From the above information, students should determine that the domain of $y=\sqrt{-x^2+6x-5}$ is $x \in [1, 5]$and the range is $y \in [0, 2]$. This should lead to the generalization that the domain of $y=\sqrt{f(x)}$ consists of all values where $f(x) \geq 0$, and the range consists of the square roots of all of the values in the range of $f(x)$ for which $f(x)$ is defined.

RF12.5 *Describe the relationship between the roots of a radical equation and the x-intercepts of the graph of the corresponding radical function.*

The relationship between the roots of a general function and the x-intercepts of the corresponding graph has been established earlier in this course (RF12). This is now extended to radical functions.

RF12.6 *Determine, graphically, an approximate solution of a radical equation.*

Initially, students should be provided with a graph of a radical function, or they should use graphing technology to create the graph, in order to approximate the solution. Students should only be expected to graph without technology radical functions that include simple transformations.

In Mathematics 2200, students solved radical equations algebraically (AN3). The intent here is to solve equations graphically.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to respond to the following:

  (i) In general, in what ways does the graph of $y = \sqrt{f(x)}$ resemble that of $y = f(x)$? In what ways does it differ?

  (RF12.3)

  (ii) Explain why the function $f(x) = \sqrt{2x+5}$ has a restricted domain while the function $f(x) = \sqrt{2x^2+5}$ has no restrictions.

  (RF12.4)

- Ask students to sketch the graphs of the radical functions below and approximate their solutions graphically:

  (i) $\sqrt{x+5} = 4$

  (ii) $\sqrt{x^2-1} = x+3$

  (RF12.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

2.2 Square Root of a Function

SB: pp. 78-89

TR: pp. 44-49

2.3 Solving Radical Equations Graphically

SB: pp. 90-98

TR: pp. 50-54

Note

Solving radical equations algebraically is a review from Mathematics 2200. It is not the focus of this outcome.

Trigonometry and the Unit Circle

Suggested Time: 12 Hours

Unit Overview

Focus and Context

In this unit, students are introduced to radian measure for angles. They also continue to work with angles in degree measure and convert between radians and degrees. They sketch angles in standard position and determine measures of coterminal angles. Students work with problems involving arc lengths, central angles, and the radius of a circle.

The equation of the unit circle, $x^2 + y^2 = 1$, is introduced and then generalized to a circle with centre (0, 0) and radius r. Students locate the coordinates on the unit circle, and identify the measure of an angle given a point on the unit circle.

Students then relate the trigonometric ratios to the coordinates of points on the unit circle. They are introduced to the reciprocal trigonometric ratios.

Students algebraically solve first-degree and second-degree trigonometric equations in radians and degrees. This will be continued throughout the remaining work with trigonometry.

Outcomes Framework

GCO

Develop trigonometric reasoning

SCO T1

Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

SCO T2

Develop and apply the equation of the unit circle.

SCO T3

Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

SCO T5

Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Trigonometry		
M4. Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. [C, CN, PS, R, T, V]	T1. Demonstrate an understanding of angles in standard position [0° to 360°]. [R, V]	T1. Demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]
M2. Apply proportional reasoning to problems that involve conversions between SI and imperial units of measure. [C, ME, PS]	T2. Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V]	T2. Develop and apply the equation of the unit circle. [CN, R, V]
	T3. Solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]	T3. Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]
		T5. Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

Trigonometry

Outcomes

Students will be expected to

T1 Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

[**CN, ME, R, V**]

Achievement Indicators:

T1.1 *Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.*

T1.2 *Sketch, in standard position, an angle with a measure of 1 radian.*

T1.3 *Describe the relationship between radian measure and degree measure.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students were introduced to angles in standard position between 0° and 360°(T1). This will be their first exposure to negative rotational angles and to rotational angles greater than 360°.

Up to this point, students have only worked with angles in degree measure. They will be introduced to radian measure and explore the relationship between radians and degrees. They will also be introduced to coterminal angles and arc length.

Revisit sketching positive angles given in degree measure. A review of terminology such as initial arm, terminal arm, vertex, sector and standard position may be required. Students should then be introduced to clockwise versus counterclockwise as it pertains to angle rotations from standard position. Some students may have difficulty equating counterclockwise with positive and clockwise with negative. It may be helpful to suggest that a positive rotation opens upward from standard position, whereas a negative angle opens downward. The *x*-axis could also be considered the “west(−) - east(+)” line and the *y*-axis the “south(−) - north(+)” line. Counterclockwise rotation from the *x*-axis can then be considered naturally positive.

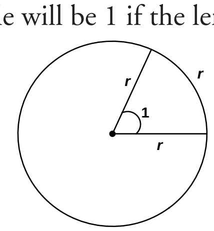
To introduce students to radian measure, they could picture a 90° angle in standard position in the coordinate plane. This angle subtends an arc equal to one-fourth the circumference of any circle centred at the origin. Students should observe that:

- In a unit circle with radius 1, a 90° angle subtends an arc $\frac{\pi}{2}$ units long since C = 2π and $\frac{1}{4}$ of 2π is $\frac{\pi}{2}$.
- In a circle of radius 5, the circumference is 10π, so a 90° angle subtends an arc length of $\frac{10\pi}{4} = \frac{5\pi}{2}$.

Students should examine the ratio of the arc length subtended by a 90° angle to the radius of the circle for various size circles. They should conclude that these ratios are all equal to $\frac{\pi}{2}$. The fact that the ratio is constant is the basis for the radian measure for angles.

$$\text{radian measure} = \frac{\text{arc length}}{\text{radius}}$$

Since the radian measure of an angle tells how many times the circle’s radius is contained in the length of the subtended arc, students should conclude that the radian measure of an angle will be 1 if the length of the subtended arc is equal to the radius.



General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Observation

- Ask students to draw a circle and show approximate angle measures of 1 radian, 2 radians, 3 radians, 4 radians, 5 radians, and 6 radians.

  (T1.2)

Interview

- Ask students to explain how to determine, with the aid of a diagram, whether 2.8 radians or 180° is a greater measure.

  (T1.3)

- Ask students to explain which is greater: 4.2 radians or 1.4π radians.

  (T1.3)

Journal

- Ask students to respond to the following:

  Your classmate has missed the introduction to radian measure. Describe, with the aid of a diagram, how to sketch an angle with a measure of 2 radians.

  (T1.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.1 Angles and Angle Measure

Student Book (SB): pp. 166-179

Teacher's Resource (TR): pp. 92-96

Trigonometry

Outcomes

Students will be expected to

T1 Continued ...

Achievement Indicators:

T1.3 *Continued*

T1.4 *Sketch, in standard position, an angle with a measure expressed in the form kπ radians, where k ∊ Q.*

T1.5 *Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.*

T1.6 *Express the measure of an angle in degrees (exact value or decimal approximation), given its measure in radians.*

Elaborations—Strategies for Learning and Teaching

After an introduction to radian measure using a circle of radius 1 unit, students should understand that 360° is equivalent to 2π in radian measure. Therefore, 180° is equivalent to π. There are certain radian measures that occur frequently. Students should become familiar with $2\pi, \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ and also with the multiples of each.

When sketching an angle with a measure of $k\pi$ radians, students should visualize angles as a fraction of π or 2π. For example, since π is equivalent to half a rotation and $\frac{\pi}{2}$ is $\frac{1}{2}$ of π, $\frac{\pi}{2}$ is a quarter of a rotation. For students having trouble visualizing angles in radian measure, it could be suggested to convert to degree measure to check the accuracy of their sketch. The goal, however, is to work with radians without having to convert to degree measure first.

Once the relationship π = 180° has been developed, students can convert between degree and radian measure. To convert from radians to degrees, they can solve this equation in terms of radians.

$$\pi \text{ rad} = 180°$$

$$\therefore 1 \text{ rad} = \frac{180°}{\pi}$$

To rewrite $\frac{9\pi}{2}$ radians as degrees, for example, they multiply by 1 radian, or $\frac{180°}{\pi}$.

Similarly, to convert to radians they should solve the equation in terms of degrees: $180° = \pi$ rad, so $1° = \frac{\pi}{180}$. Students should become comfortable expressing radian measure in both exact and approximate values.

A common error occurs when students invert the conversion factor. This may be avoided through unit analysis which was worked with in Mathematics 1201 (M2).

Students should note that any angle measure given without a degree symbol is assumed to be in radians.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to sketch the following angles in standard position:
  (i) $\frac{\pi}{4}$
  (ii) $\frac{2\pi}{3}$
  (iii) $\frac{7\pi}{6}$
  (iv) $-\frac{3\pi}{2}$

(T1.4)

- Ask students to express the measures of the following angles in radian measure:
  (i) 60°
  (ii) 150°
  (iii) −225°
  (iv) −144°
  (v) 214.5°

(T1.5)

- Ask students to express the following radian measures in degrees:
  (i) $\frac{2\pi}{3}$
  (ii) $-\frac{7\pi}{4}$
  (iii) $\frac{11\pi}{12}$
  (iv) −5

(T1.6)

Performance

- For *Five-Minute Review*, allow teams five minutes to review how to sketch angles in standard position and the relationship between degrees and radians. Students in their groups can ask a clarifying question to the other members or answer questions of others.

(T1.1, T1.4, T1.5, T1.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.1 Angles and Angle Measure

SB: pp. 166-179

TR: pp. 92-96

Trigonometry

Outcomes

Students will be expected to

T1 Continued ...

Achievement Indicators:

T1.7 *Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.*

T1.8 *Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.*

T1.9 *Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius r, and solve problems based upon that relationship.*

Elaborations—Strategies for Learning and Teaching

The concept of coterminal angles is new to students. Coterminal angles are angles in standard position with the same terminal arms and can be measured in degrees or radians. Examples should include both positive and negative coterminal angles, found by adding or subtracting multiples of 360° or 2π. This leads to developing the general form, expressed as $\theta \pm (360°)n$ or $\theta \pm 2\pi n$ where $n \in$ W. The general solution could also be expressed as $\theta + (360°)n$ or $\theta + 2\pi n$, $n \in$ I. When introducing coterminal angles, encourage students to sketch the angles. This highlights the fact that coterminal angles share a terminal arm.

Based on the definition of a radian, the relationship between a central angle θ and the length of the arc cut on a circle of radius *r* can be developed.

arc length = $\theta \times$ radius, where θ is the central angle in radians

Students should be able to determine any variable in the relationship, given the measure of the other two. Given an arc length of 20 cm cut on a circle of radius 5.4 cm, for example, they could be asked to determine the measure of the central angle in radians or degrees. They have the choice of rearranging the equation in terms of θ first, or substituting the known values before solving for θ.

$$\text{arc length} = \theta \times \text{radius}$$

$$20 = \theta \times 5.4$$

$$\theta \approx 3.7 \text{ radians or } 212°$$

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine a positive and a negative angle measure that is coterminal to the following measures:

  (i) 120°

  (ii) −515°

  (iii) $\frac{3\pi}{4}$

  (iv) -9π

  (T1.7)

- Ask students to determine the measure of all angles that are coterminal with:

  (i) 310°

  (ii) $-\frac{3\pi}{4}$

  (T1.8)

- Ask students to determine the measures of the arc length subtended by the angles and radii below:

  (i) Central angle of $\frac{2\pi}{3}$ with radius 10 cm

  (ii) Central angle of 2.6 rad with radius 4.9 cm

  (T1.9)

- Ask students to determine the measure of the radius of a circle if an arc length of 42 ft. is subtended by an angle of $\frac{7\pi}{4}$.

  (T1.9)

- Ask students to answer the following:

  During a family vacation, you go to dinner at the Seattle Space Needle. There is a rotating restaurant at the top of the needle that is circular and has a radius of 40 feet. It makes one rotation per hour. At 6:42 p.m., you take a seat at a window table. You finish dinner at 8:28 p.m. Through what angle did your position rotate during your stay? How many feet did your position revolve?

  (T1.9)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.1 Angles and Angle Measure

SB: pp. 166-179

TR: pp. 92-96

Trigonometry

Outcomes

Students will be expected to

T2 Develop and apply the equation of the unit circle.

[CN, R, V]

Achievement Indicators:

T2.1 *Derive the equation of the unit circle from the Pythagorean theorem.*

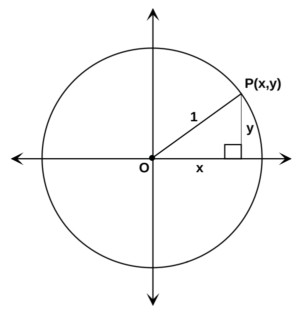
T2.2 *Generalize the equation of a circle with centre (0,0) and radius r.*

T2.3 *Describe the six trigonometric ratios, using a point P(x, y) that is the intersection of the terminal arm of an angle and the unit circle.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students determined the exact values of the primary trigonometric ratios for angles between 0° and 360° using the relationship between the sides of special right triangles. Students may have been introduced to the unit circle as a strategy for finding exact values; however, it was not a direct outcome. Now students will be formally introduced to the unit circle and use its equation to generalize the equation of any circle centred at the origin.

Students can find the equation of the unit circle using the Pythagorean theorem. Using a circle of radius 1 unit centred at the origin, they should mark a point P on the circle and draw a right triangle. Ask them to consider why the absolute value of the *y*-coordinate represents the distance from a point to the *x*-axis.



Applying the Pythagorean theorem results in:

$$x^2 + y^2 = 1^2$$

$$x^2 + y^2 = 1$$

Ask students how the equation would differ if the radius was *r* instead of 1. From this, they should generalize the equation of a circle with centre (0,0) and radius *r* to be $x^2 + y^2 = r^2$.

Given an angle θ in standard position, expressed in degrees or radians, students should determine the coordinates of the corresponding point on the unit circle. Conversely, they should determine an angle in standard position that corresponds to a given point on the unit circle.

In Mathematics 1201 and 2200, students worked with the three primary trigonometric ratios. This is their first exposure to the reciprocal ratios: csc θ, sec θ and cot θ.

Revisiting the unit circle above, students should observe that $\sin\theta = \frac{y}{1}$, $\cos\theta = \frac{x}{1}$, and $\tan\theta = \frac{y}{x}$. They should also see that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Once students have been introduced to the reciprocal ratios as $\csc\theta = \frac{1}{y}$, $\sec\theta = \frac{1}{x}$, and $\cot\theta = \frac{x}{y}$, it follows as a natural extension to discuss the following relationships:

$\csc\theta = \frac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}$ $\qquad$ $\cot\theta = \frac{1}{\tan\theta}$ or $\frac{\cos\theta}{\sin\theta}$

Students should be exposed to the restrictions on θ for tan θ, csc θ, sec θ and cot θ.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the equation of a circle whose centre is at (0, 0) with a radius of:
  (i) 4 units
  (ii) $\sqrt{34}$ units
  (iii) $2\sqrt{5}$ units

(T2.2)

- Ask students to determine whether the given point is on the circle whose equation is given.
  (i) P(−2, 6); $x^2 + y^2 = 49$
  (ii) $P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; the unit circle

(T2.2)

- Ask students to determine the *x*- or *y*- coordinate on the unit circle given the other coordinate. Use examples such as:
  (i) $P\left(\frac{1}{5}, y\right)$ in Quadrant I
  (ii) $P\left(x, -\frac{3}{8}\right)$ in Quadrant III

(T2.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.2 The Unit Circle

SB: pp. 180-190

TR: pp. 97-101

4.3 Trigonometric Ratios

SB: pp. 191-205

TR: pp. 102-107

Trigonometry

Outcomes

Students will be expected to

T3 Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

[ME, PS, R, T, V]

Achievement Indicators:

T3.1 *Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either degrees or radians.*

T3.2 *Determine, using a unit circle or reference triangle, the exact value of a trigonometric ratio for angles expressed in degrees that are multiples of 0°, 30°, 45°, 60° or 90°, or for angles expressed in radians that are multiples of 0,* $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, *or* $\frac{\pi}{2}$, *and explain the strategy.*

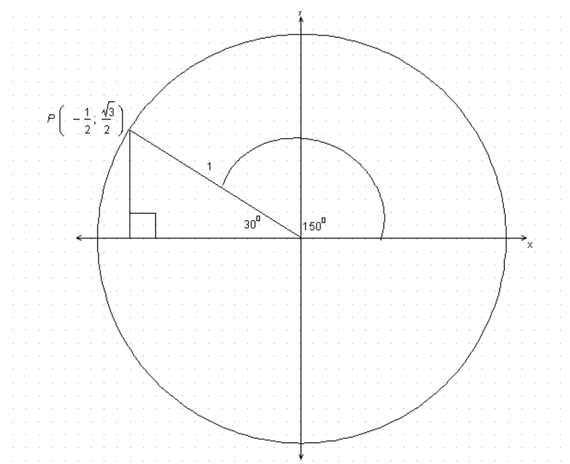
T3.3 *Sketch a diagram to represent a problem that involves trigonometric ratios.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students solved problems using the three primary trigonometric ratios (T2). Angles were expressed in degrees. This will now be extended to include the reciprocal ratios. Students will work with angles expressed in both degrees and radians.

Students should understand how to properly use a scientific or graphing calculator to evaluate all six of the trigonometric ratios. They should be able to efficiently use their calculators in both degree and radian mode, being careful to check for the appropriate mode in all calculations.

Students have had previous experience determining reference angles for positive rotational angles. This can also be applied to negative rotational angles. Students can also use their previous knowledge of reference triangles and/or the unit circle to determine the exact value of the six trigonometric ratios. From the sketch below, for example, they can determine the value of the trigonometric ratios for 150°.



$$\cot 150° = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3}$$

Students should also simplify expressions such as $\dfrac{\cos\left(\frac{\pi}{6}\right)+\sin(-\pi)}{\tan(30°)}$.

Expressions requiring rationalizing are limited to those with monomial denominators. From Mathematics 2200, students are familiar with performing operations on rational expressions (AN5).

Students could use a calculator to verify their answers. The emphasis here, however, is on finding exact values using the unit circle, reference triangles, and mental math strategies.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to find the exact value of the following expressions:

  (i) $\sin\left(-\frac{2\pi}{3}\right)+\cos^2\left(\frac{11\pi}{6}\right)$

  (ii) $\csc\left(\frac{\pi}{3}\right)+\cot\left(\frac{11\pi}{4}\right)$

  (iii) $\cot^2\left(-\frac{7\pi}{6}\right)$

  (iv) $\dfrac{\cot\left(\frac{\pi}{3}\right)+\cos\left(\frac{11\pi}{3}\right)}{\csc\left(-240°\right)}$

  (T3.2, T3.3)

Performance

- For the *Four Corners* activity, each student is given a card containing a different trigonometric ratio. The four corrners of the room correspond to the four quadrants. Students decide which quadrant the angle on the card lies in and they move to the appropriate corner. The small group should then verify that the angles are appropriately placed in the quadrant, and evaluate each of the trigonoemtric ratios.

  Cards should include a mixture of angles in degree and radian measure, positive and negative angles, and those that have exact and approximate values.

  Sample Ratios:

$\cos\frac{4\pi}{3}$	$\tan\left(-210°\right)$	$\csc\frac{11\pi}{4}$	$\sin\left(-310°\right)$
$\cot\frac{5\pi}{3}$	$\sec 315°$	$\sin 123°$	$\sec 3.6$

  (T3.1, T3.2, T3.3)

- In groups, students can play the Carousel game. Stations are set up around the room, with completed expressions posted on the wall. Students identify the error(s) and write the correct solution.

  (T3.2, T3.3)

- Students can play a Match game. One set of cards contains trigonometric expressions and the second set contains the completed solutions. Teachers can vary the set-up, whereby they can have the cards face up or face down.

  (T3.2, T3.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.3 Trigonometric Ratios

SB: pp. 191-205

TR: pp. 102-107

Note

Simplifying trigonometric expressions involving rationalizing requires supplementary practice questions.

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

Achievement Indicators:

T3.4 *Determine, with or without technology, the measures, in degrees or radians, of the angles in a specified domain, given the value of a trigonometric ratio.*

T3.5 *Describe the six trigonometric ratios, using a point P(x, y) that is the intersection of the terminal arm of an angle and the unit circle.*

T3.6 *Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.*

Elaborations—Strategies for Learning and Teaching

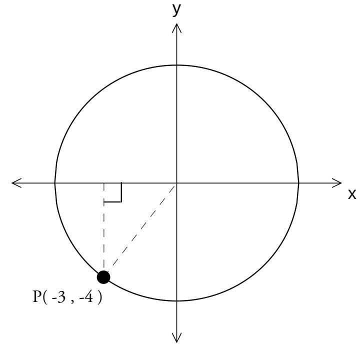
Students determined the measures of angles in degrees for the three primary trigonometric ratios. They have also used reference triangles and the unit circle to evaluate the six trigonometric ratios. Determining the measure of angles for all six trigonometric ratios in both radians and degrees should be a natural extension. Students could be asked to determine the value of θ, for example, when $\sec\theta = -\sqrt{2}$ for the domain $-2\pi \le \theta \le 2\pi$. To determine the reference angle, they could think about a triangle with hypotenuse $\sqrt{2}$ and adjacent side 1. Alternatively, they could apply the reciprocal ratio $\cos\theta = -\frac{1}{\sqrt{2}}$. Once the reference angle is determined, they identify the quadrants where the secant ratio is negative.

QII:	QIII:
$\pi - \frac{\pi}{4}$	$\pi + \frac{\pi}{4}$
$\frac{3\pi}{4}$	$\frac{5\pi}{4}$

The final step focuses on identifying all possible values within the given domain. Remind students of earlier work with coterminal angles.

$$\theta = \left\{-\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}\right\}$$

Given the coordinates of a point on the terminal arm of an angle in standard position, the six trigonometric ratios can be determined. To determine the ratios given that the point P(−3, −4) lies on the terminal arm of the angle, for example, students should first sketch a diagram. Point out that the right triangle is always made with the *x*-axis.



Encourage them to note that because the angle is in the third quadrant, sine, cosine and their reciprocals are negative, and tangent and cotangent are positive. This can be used to check the reasonableness of their ratios. Students should also be asked to determine the measure of the given angle in radians or degrees.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve the following trigonometric equations:

  (i) $\sin\theta = -\frac{\sqrt{3}}{2}$; $0 \le \theta < 2\pi$
  (ii) $\csc\theta = -2$; $0^\circ \le \theta < 360^\circ$
  (iii) $\sec\theta = -\frac{2\sqrt{3}}{3}$; $-2\pi \le \theta < 2\pi$

(T3.4)

- Given that each of the following points lie at the intersection of the unit circle and the terminal arm of an angle in standard position, ask students to:
  (i) sketch the diagram
  (ii) determine the values of the six trigonometric ratios
  (iii) determine the angle of rotation from standard position
  (a) $P\left(\frac{5}{13}, -\frac{12}{13}\right)$

  (b) $P\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

  (c) $P(-5, 4)$ (T3.5, T3.6)

Journal

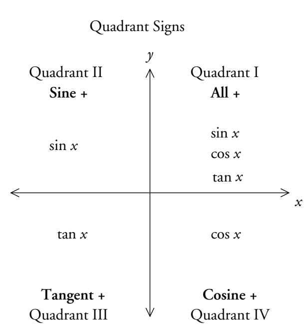
- Ask students to respond to the following:

  Give an example of a trigonometric equation that does not have a solution. Explain why.

(T3.4)

Performance

- Ask students to summarize what they have learned about the location of positive and negative trigonometric ratios in the four quadrants. They should use a graphic organizer such as:



Students could also include the reciprocal ratios.

(T3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.3 Trigonometric Ratios

SB: pp. 191-205

TR: pp. 102-107

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

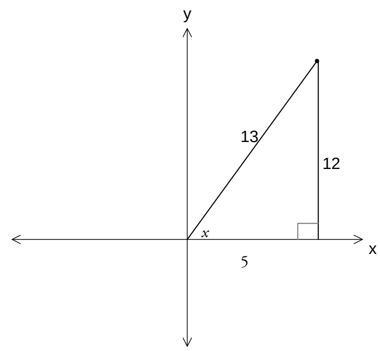
Achievement Indicators:

T3.7 *Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.*

T3.8 *Solve a problem, using trigonometric ratios.*

Elaborations—Strategies for Learning and Teaching

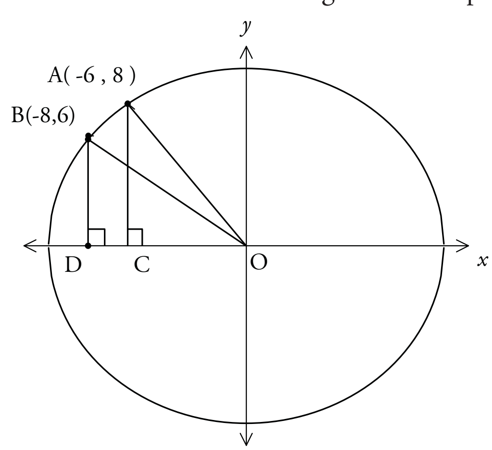
Students have had previous experience finding the other two primary trigonometric ratios given one of them. This is now extended to include the reciprocal trigonometric ratios. Given $\cos x = \frac{5}{13}$, for example, they could be asked to determine the value of csc x for $0° \leq x \leq 90°$. They should be encouraged to draw a sketch similar to the one below to help visualize a solution.



Students should be exposed to cases with a variety of domains, including negative values.

Thorough understanding of the trigonometric ratios and work with angles in standard position (T1) could be determined by asking students to find the distance between two given points on a circle. Consider using an example such as the following:

Students could be asked to find the arc length between points A(−6, 8) and B(−8,6).



Using A(−6,8): $\tan\angle AOC = \frac{8}{6}$ $\qquad \therefore \angle AOD \doteq 53°$

Using B(−8, 6): $\tan\angle BOD = \frac{6}{8}$ $\qquad \therefore \angle BOD \doteq 37°$

This results in $\angle AOB \doteq 16°$, and thus a curved arc length between points A and B of 2.8 units.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Students could answer questions such as:

  (i) Given $\sin\theta = -\frac{1}{2}$, $180° \leq \theta \leq 270°$, determine the value of $\cot\theta$.

  (ii) Given $\sec x = 5.5$, where $\frac{3\pi}{2} \leq x \leq 2\pi$, determine $\tan x$.

  (T3.7)

  (iii) A regular dodecagon (12-sided figure) is inscribed in the unit circle. If one vertex is at (1,0), what are the exact coordinates of the other vertices? Explain your reasoning.

  (T3.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.3 Trigonometric Ratios

SB: pp. 191-205

TR: pp. 102-107

Trigonometry

Outcomes

Students will be expected to

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

[CN, PS, R, T, V]

Achievement Indicators:

T5.1 *Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible.*

T5.2 *Determine, using technology, the approximate solution of a trigonometric equation.*

T5.3 *Verify, with or without technology, that a given value is a solution to a trigonometric equation.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students solved simple trigonometric equations of the form sin θ = *a* or cos θ = *a*, where $-1 \leq a \leq 1$, and tan θ = *a*, where *a* is a real number (T2). They worked with angles in degree measure. This will now be extended to include trigonometric equations with all six trigonometric ratios. Students will solve first and second degree trigonometric equations with the domain expressed in degrees and radians.

When solving first degree equations, rearrangement will sometimes be necessary to isolate the trigonometric ratio. When solving an equation such as sec θ = $\sqrt{2}$, students can consider which reference angle results when the hypotenuse is $\sqrt{2}$ and the adjacent side is 1. Alternatively, they can think about secant as the reciprocal of cosine and determine the reference angle for the equation cos θ = $\frac{1}{\sqrt{2}}$. Knowledge of exact values of the sine, cosine or tangent of a 30°, 45° or 60° angle, as well as their corresponding radian measures, and an understanding of the unit circle continue to be important when solving trigonometric equations.

Students also solve second degree equations through techniques such as factoring (e.g., $\sin^2\theta - 3\sin\theta + 2 = 0$, for all θ) or isolation and square root principles (e.g., $\tan^2\theta - 3 = 0$, $0 \leq \theta < 2\pi$). Students sometimes mistakenly use only the principal square root when both negative and positive should be considered. In the equation $\tan^2\theta - 3 = 0$, $0 \leq \theta < 2\pi$, isolating the trigonometric ratio results in tan θ = $\pm\sqrt{3}$ Students should realize that using only the principal square root in this equation causes a loss of roots. Another common error occurs when students do not find all solutions for the given domain. Remind them to focus on the given domain. In the example above, the reference angle is $\frac{\pi}{3}$ and since there are two cases to consider (tangent being negative and positive), there are solutions in all four quadrants.

Students should be encouraged to check all solutions with a calculator or using the unit circle where appropriate. When solving equations containing reciprocal ratios, students should also check that the solutions are defined for the domain of the reciprocal ratios.

In the next unit, Trigonometric Functions and Graphs, students will solve trigonometric equations for which the argument may include a horizontal stretch or horizontal translation, such as $\cos(2\theta - \pi) = -\frac{1}{2}$.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve the following trigonometric equations:

  (i) $\sqrt{2}\cos x - 1 = 0;\ x \in [-2\pi, 2\pi]$
  (ii) $4\cot\theta + 3 = -2\cot\theta - 8;\ \theta \in (0, 360°)$
  (iii) $2\csc^2\theta - 8 = 0$; for all θ in radians.
  (iv) $5\sec^2 x = 1 - \sec x$; for all x in radians.
  (v) $2\sin^2 x + 5\sin x - 3 = 0;\ x \in (-\pi, 2\pi)$

  (T5.1, T5.2)

- Given $g(\theta) = \cos^2\theta - 3$ and $p(\theta) = 2\cos\theta$, ask students to determine the values of θ such that $g(\theta) = p(\theta)$, where $\theta \in [0, 4\pi]$.

  (T5.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.4 Introduction to Trigonometric Equations

SB: pp. 206-214

TR: pp. 108-111

Trigonometry

Outcomes

Students will be expected to

T5 Continued ...

Achievement Indicator:

T5.4 *Identify and correct errors in a solution for a trigonometric equation.*

Elaborations—Strategies for Learning and Teaching

Students have had exposure to identifying and correcting errors throughout the mathematics curriculum. This approach is continued in the context of solving trigonometric equations.

Students should be encouraged to check the entire given solution for errors and not stop checking once they have encountered the first error. The following solution, for example, contains two errors: the first error stems from improperly applying the zero product principle and the second error occurs when all solutions for the given domain are not found.

$$2\cos^2\theta - \cos\theta - 1 = 0,\quad 0 \le \theta < 360^\circ$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$$\cos\theta = \tfrac{1}{2},\quad \cos\theta = -1$$
$$\theta = \cos^{-1}\left(\tfrac{1}{2}\right),\quad \theta = \cos^{-1}(-1)$$
$$\therefore \theta = 60^\circ, 180^\circ$$

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Observation

- Set up centres containing examples of trigonometric equations that have been solved incorrectly. Students should move around the centres to identify and correct the errors. Samples are shown below:

(i) $4\sin^2 x - 2 = 0$, for all values of x in radians

$\sin^2 x = \frac{1}{2}$

$\sin x = \sqrt{\frac{1}{2}}$

$\sin x = \frac{\sqrt{2}}{2}$

ref $\angle = \frac{\pi}{4}$

Quadrant I: $\frac{\pi}{4}$

Quadrant II: $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$x = \begin{cases} \frac{\pi}{4} + 2\pi n, \ n \in I \\ \frac{3\pi}{4} + 2\pi n, \ n \in I \end{cases}$$

(ii) $2\tan^2 x - 5\tan x - 3 = 0, \ -360° \le x \le 360°$

$(\tan x - 3)(2\tan x + 1) = 0$

$\tan x = 3$	$\tan x = -\frac{1}{2}$
ref $\angle \doteq 71.6°$	ref $\angle \doteq 26.6°$
Quadrant I: $71.6°$	Quadrant II: $153.4°$
Quadrant III: $251.6°$	Quadrant IV: $333.4°$

$x = 71.6°, 153.4°, 251.6°, 333.4°$

(T5.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

4.4 Introduction to Trigonometric Equations

SB: pp. 206-214

TR: pp. 108-111

Trigonometric Functions and Graphs

Suggested Time: 12 Hours

Unit Overview

Focus and Context

In this unit, students sketch the graphs of $y = \sin x$ and $y = \cos x$ and determine characteristics such as the period, amplitude, maximum and minimum values, intercepts, and domain and range. They also explore the effects of transformations on these graphs.

Once each of these functions has been explored, trigonometric functions are determined to model real-world situations and solve problems.

Students sketch the graph of $y = \tan x$ and identify the domain and range, period, asymptotes and intercepts.

Outcomes Framework

GCO

Develop trigonometric reasoning.

SCO T4

Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

SCO T5

Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Trigonometry		
M4. Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. [C, CN, PS, R, T, V]	T1. Demonstrate an understanding of angles in standard position [0° to 360°]. [R, V] T2. Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V] T3. Solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]	T4. Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS., T, V] T5. Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

Trigonometry

Outcomes

Students will be expected to

T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

[CN, PS, T, V]

Achievement Indicators:

T4.1 *Sketch, with or without technology, the graphs of $y = \sin x$ and $y = \cos x$.*

T4.2 *Determine the characteristics (amplitude, domain, period, range and zeros) of the graphs of $y = \sin x$ and $y = \cos x$.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201 students first encountered and applied the three primary trigonometric ratios (M4). In Mathematics 2200, students developed an understanding of angles as rotations, and solved simple trigonometric equations in degrees only. They also graphed reciprocal functions (limited to the reciprocal of linear and quadratic functions) and developed an understanding of asymptotes. Earlier in Mathematics 3200, students were exposed to work in radian measure, the unit circle, and solved more complex trigonometric equations (T1, T2, T3, T5).

In the previous unit, students determined the values of $\sin x$ and $\cos x$ for various angle measures in degrees and radians. They also explored the unit circle. In this unit, students develop the graphs for $y = \sin(x)$, $y = \cos(x)$, and later $y = \tan(x)$ and identify the characteristic features of each. Transformations of $y = \sin(x)$ and $y = \cos(x)$ are performed and are used to model and solve problems. Students are not expected to transform $y = \tan(x)$ or solve problems that involve the tangent function.

Students could use the values from the unit circle to plot $(\theta, \cos\theta)$ and $(\theta, \sin\theta)$ on the interval $x \in [0, 2\pi]$. These results can be effectively verified using graphing technology.

Draw attention to the 5 key points associated with each graph, since these points help determine the characteristics of the graph (amplitude, domain, period, range, and zeros) and will also be used to graph transformations of them.

$y = \sin(x)$:

0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
0 sinusoidal axis	1 maximum	0 sinusoidal axis	-1 minimum	0 sinusoidal axis

$y = \cos(x)$:

0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
1 maximum	0 sinusoidal axis	-1 minimum	0 sinusoidal axis	1 maximum

For each base graph the 5 key points give a skeleton of the graph on the interval 0 to 2π, (one complete period) with points spaced a quarter-period apart. Students should develop a proficiency in using patterning to continue the skeleton and to sketch the graph. Their graphs should show the periodic and continuous nature of the graphs of sinusoidal functions. This requires that students show more than one period of a sinusoidal graph. In addition to the characteristics listed, students should also identify local maximums and minimums, and the equation of the sinusoidal axis, also referred to as the horizontal central axis.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Journal

- Ask students to create a Venn diagram to compare the characteristics of the functions $y = \sin(x)$ and $y = \cos(x)$.

(T4.2)

Interview

- Ask students to show if the point $\left(-\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ lies on the graph of $y = \sin(x)$. Ask them if the same point lies on the graph of $y = \cos(x)$.

(T4.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

5.1 Graphing Sine and Cosine Functions

Student Book (SB): pp. 222-237

Teacher's Resource (TR): pp. 118-123

Trigonometry

Outcomes

Students will be expected to

T4 Continued ...

Achievement Indicators:

T4.3 *Determine how varying the value of a affects the graph of $y = a \sin x$ and $y = a \cos x$.*

T4.4 *Determine how varying the value of b affects the graphs of $y = \sin bx$ and $y = \cos bx$.*

T4.5 *Determine how varying the value of d affects the graphs of $y = \sin x + d$ and $y = \cos x + d$.*

T4.6 *Determine how varying the value of c affects the graphs of $y = \sin(x + c)$ and $y = \cos(x + c)$.*

T4.7 *Sketch, without technology, graphs of the form $y = a\sin b\left(x - c\right) + d$ and $y = a\cos b\left(x - c\right) + d$ using transformations, and explain the strategies.*

T4.8 *Determine the characteristics (amplitude, domain, period, phase shift, range and zeros) of the graphs of trigonometric functions of the form $y = a\sin b\left(x - c\right) + d$ and $y = a\cos b\left(x - c\right) + d$.*

Elaborations—Strategies for Learning and Teaching

Students have developed an understanding of transformations for graphs of the form $y = af(b(x - \mathrm{h})) + k$ in the Function Transformations unit, and will now apply that knowledge to sinusoidal graphs, noting that parameters h and k are now referred to as c and d. Students should determine the end position of the five key points using transformations, and then extend the graph appropriately.

Students need to know what characteristics of the graph change with each parameter. To achieve this, they should investigate how each parameter change affects the resulting graph one parameter at a time, and match these changes to the defining characteristics of sinusoidal graphs. Changing the value of a, for example, affects the amplitude of the graph, and negative values result in a reflection in the x-axis. These explorations can be effectively carried out using graphing technology and could be accomplished through a guided exploration of the parameters.

Ensure that students work with non-integer values for these parameters since many real-world applications do not use integer values. They should be exposed to negative values for a and b, but the focus of the work should be with positive values.

Students should be able to determine the characteristics of a trigonometric function using the value of the parameters in the equation without necessarily having to graph the equation. Linking students' knowledge of transformations to their understanding of the language of sinusoidal functions should be emphasized here rather than memorizing formulas. Rather than only memorizing period = $\frac{2\pi}{|b|}$, for example, the concept should be developed from an understanding that b affects the horizontal stretch, and the period of the base graph for sine is 2π.

Students should know from the Function Transformations unit that the c parameter relates to horizontal translation. With sinusoidal functions this is referred to as the phase shift, and will determine how the 5 key points will be translated horizontally.

Students should be encouraged to use appropriate mathematical language. They are expected to understand the terms phase shift and vertical displacement, recognizing that they refer to horizontal and vertical translations, respectively.

Students should also analyze sinusoidal functions that may require factoring. The function $y = \sin(4x - 6\pi)$, for example, should be factored to $y = \sin 4\left(x - \frac{3}{2}\pi\right)$ to correctly identify the phase shift.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Performance

- For the function $y = a \sin b(x - c) + d$, the parameters a, b, c, and d will be randomly chosen (cards, dice, etc). Students compete in teams to be the first to identify the five key points.

  Alternatively, they could be asked to determine the period or any of the other characteristics.

  (T4.3 to T4.8)

- For a *Jigsaw* activity, students begin in a home group and each member is a given a number (1 to 4). The groups dissolve and the students with the same numbers form expert groups. Each expert group explores a different type of transformation for $y = \sin x$ and $y = \cos x$. Once the exploration is complete, students return to their home groups. Each expert teaches the others about the transformation they explored.

  (T4.3 to T4.6)

Paper and Pencil

- Students are each given the equations of two sinusoidal functions (e.g., $y = 3\sin(2(x – 30°)) + 6$ and $y = -4\cos(2(x – 60°)) + 6$) and asked to describe how the functions are alike and how they differ.

  (T4.3 to T4.8)

Interview

- Ask students the following questions about the curve $y = 5\cos 3(x - 30°) + 2$:
  - (i) What is the period?
  - (ii) What is the amplitude?
  - (iii) What is the range?
  - (iv) Suppose we wanted to write the equation in the form $y = a \sin b(x - c) + d$. What values could be used for a, b, c and d?
  - (v) How could the equation be modified so that the resulting function will have no x-intercepts?

  (T4.3 to T4.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

5.1 Graphing Sine and Cosine Functions

SB: pp. 222-237

TR: pp. 118-123

Web Link

www.desmos.com

This online graphing calculator can be used to graph functions, plot tables of data, evaluate equations, and explore transformations.

5.2 Transformations of Sinusoidal Functions

SB: pp. 238-255

TR: pp. 124-129

Trigonometry

Outcomes

Students will be expected to

T4 Continued ...

Achievement Indicators:

T4.9 *Determine the values of a, b, c and d for functions of the form y = a sin b(x - c) + d and y = a cos b(x - c) + d that correspond to a given graph, and write the equation of the function.*

T4.10 *Solve a given problem by analyzing the graph of a trigonometric function.*

T4.11 *Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.*

T4.12 *Determine a trigonometric function that models a situation to solve a problem.*

Elaborations—Strategies for Learning and Teaching

To determine the equation of a sinusoidal function, students will have to determine the key characteristics of the graph, and then link them to the parameters in the equation. If they determine that the amplitude for the graph is 4, for example, they should realize that $a = 4$.

While there are an infinite number of correct choices for the phase shift, the convention is to use the smallest positive value. Teachers need to be careful to accept all correct answers, not just the conventional answer when assessing student work. Students should also be made aware that more than one correct equation is possible and they should be able to identify equations that produce the same graph.

Students should be cautioned when using negative values for *a* and *b*, since these would involve reflections. Consequently, when determining the value of the phase shift, different points on the graph need to be considered.

Trigonometric functions are commonly used to model problems that are periodic in nature, including circular motion, pistons, tides, climate, daylight, populations of species, electricity, etc.

Students should be encouraged to draw well-labelled sketches of the graphs that represent the problems. This will help them more easily identify the characteristics of the trigonometric function. While students should be able to write a sine and a cosine equation to model any application, cosine is more often used since it is easier to identify a correct phase shift. Students are free to choose any correct equation to solve the problem.

Teachers need to be mindful of the type of assessment questions used. Questions that require graphing technology should be explored in a classroom setting, not on formal assessments.

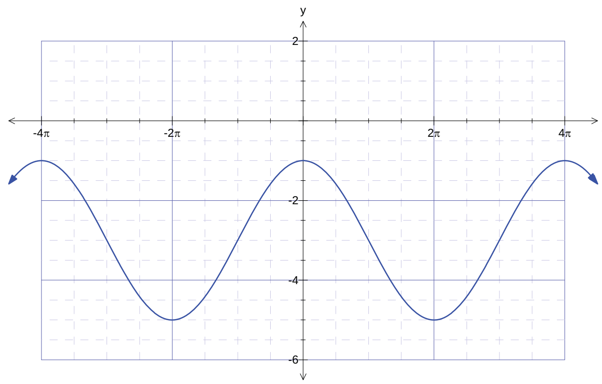
General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Journal

- Students could be given the graph and the equation of a sinusoidal function. Ask them to determine if the equation is correct for the graph. If not, they should explain how the equation would need to be modified.

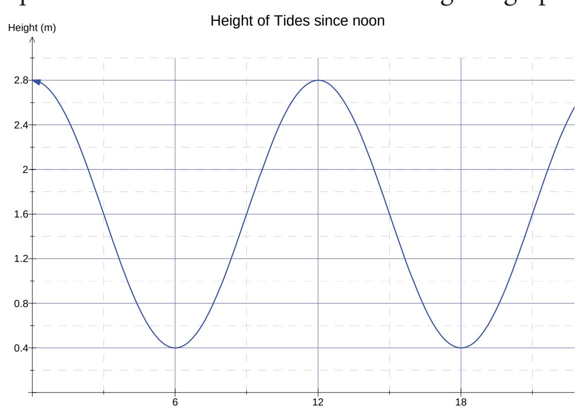
  Example:



$$y = 2\cos(2x) - 3$$

(T4.8, T4.9)

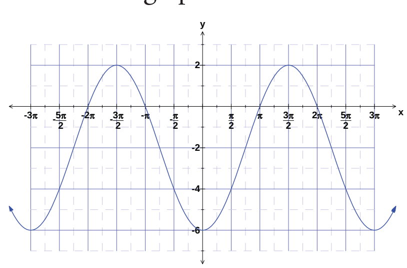
- Students could be given a graph that models a problem and asked to identify questions that could be answered using the graph.



(T4.10, T4.11)

Paper and Pencil

- Ask students to determine the values of *a*, *b*, *c*, and *d* required to write an equation for the graph.



(T4.8, T4.9)

Resources/Notes

Authorized Resource

Pre-Calculus 12

5.2 Transformations of Sinusoidal Functions

SB: pp. 238-255

TR: pp. 124-129

Trigonometry

Outcomes

Students will be expected to

T4 Continued ...

Achievement Indicators:

T4.13 *Sketch, with or without technology, the graph of y = tan x.*

T4.14 *Determine the characteristics (asymptotes, domain, period, range and zeros) of the graph of y = tan x.*

Elaborations—Strategies for Learning and Teaching

Although students have been exposed to the concept of a tangent line in Grade 9, they may need reminding before investigating the graph of $y = \tan x$.

Students should develop the graph of $y = \tan(x)$, paying attention to the characteristics of this graph. As with sine and cosine, 5 key points (three points and two asymptotes) are conventionally used to graph one complete period of $y = \tan(x)$:

$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
asymptote	-1	0	1	asymptote

As with sine and cosine, ensure students realize that these key points help guide the construction of one period of the function and they should be able to use these key points to produce reasonably accurate graphs. The graphs should be extended to include more than one period.

Discuss whether the graph of $y = \tan(x)$ has an amplitude. It may be helpful for students to see that “amplitude” is a characteristic of sine and cosine graphs and it depends on a maximum and a minimum height. Since the function $y = \tan(x)$ has no maximum or minimum, it cannot have an amplitude.

A similar discussion could be held to help students understand why the period of $y = \tan(x)$ is π.

Note that transformations of $y = \tan(x)$ will not be explored in this course.

When discussing the behaviour around the asymptotes, it is important to remember that students have no formal experience with limits, limit notation, or infinity. Ask students to analyse the slope of the terminal arm on the unit circle as it rotates from 0° to 90°.

- When is the slope 0?
- What happens as the angle increases?
- What happens to the slope as the angle approaches 90°?
- What is the slope of the line at 90°?

Ask students the same questions concerning rotations from 0° to −90°. Have them compare these to slopes when the terminal arm is rotated past 90°. They should see the periodic nature of the curve and the behaviour around the asymptotes.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to sketch a graph of $y = \sin(x)$ and $y = \tan(x)$ on the same grid.

  They should explain how the x-intercepts of $y = \sin(x)$ relate to the x-intercepts of $y = \tan(x)$.

  (T4.13, T4.14)

- Ask students to sketch the graph of $y = \cos(x)$ and $y = \tan(x)$ on the same grid.

  They should explain how the x-intercepts of $y = \cos(x)$ relate to the asymptotes of $y = \tan(x)$.

  (T4.13, T4.14)

Interview

- Ask students if $x = 3\pi$ is part of the domain of $y = \tan(x)$.

  (T4.14)

Resources/Notes

Authorized Resource

Pre-Calculus 12

5.3 The Tangent Function

SB: pp. 256-265

TR: pp. 130-133

Note

SB: p. 263, #5-#7

These questions develop the relationship between $\tan x = \frac{\sin x}{\cos x}$ and the slope of the terminal arm, which connects nicely with the Trigonometry and the Unit Circle unit.

Trigonometry

Outcomes

Students will be expected to

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

[CN, PS, R, T, V]

Achievement Indicators:

T5.1 *Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible.*

T5.2 *Determine, using technology, the approximate solution of a trigonometric equation.*

T5.3 *Verify, with or without technology, that a given value is a solution to a trigonometric equation.*

T5.5 *Relate the general solution of a trigonometric equation to the zeros of the corresponding function (restricted to sine and cosine functions).*

Elaborations—Strategies for Learning and Teaching

In Mathematics 2200, students solved quadratic equations by graphing the related quadratic function and determining its *x*-intercepts (RF3, RF4). They also solved systems of equations (quadratic-quadratic and quadratic-linear) graphically and algebraically (RF6).

In the previous unit, students solved first and second degree trigonometric equations algebraically. In this unit, they continue to solve equations algebraically. They also use the graphs of trigonometric functions to solve equations. For assessment purposes, students should analyze given graphs to determine solutions.

Students also solve trigonometric equations for which the argument may include a horizontal stretch or a horizontal translation.

To algebraically determine all solutions in radian measure for $\cos\left[4\left(x-\frac{\pi}{2}\right)\right]=-\frac{\sqrt{3}}{2}$ students could proceed as follows:

$$\text{Ref }\angle : \theta_R = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$$

$$\text{Quadrant II: } \frac{5\pi}{6}$$

$$\text{Quadrant III: } \frac{7\pi}{6}$$

$$\therefore 4\left(x-\frac{\pi}{2}\right)=\begin{cases}\frac{5\pi}{6}+2\pi n,\ n\in I\\ \frac{7\pi}{6}+2\pi n,\ n\in I\end{cases}$$

$$x-\frac{\pi}{2}=\begin{cases}\frac{5\pi}{24}+\frac{\pi}{2}n,\ n\in I\\ \frac{7\pi}{24}+\frac{\pi}{2}n,\ n\in I\end{cases}$$

$$x=\begin{cases}\frac{17\pi}{4}+\frac{\pi}{2}n,\ n\in I\\ \frac{19\pi}{4}+\frac{\pi}{2}n,\ n\in I\end{cases}$$

Students should be reminded to check solutions by evaluating each side of the original equation at all the solution values.

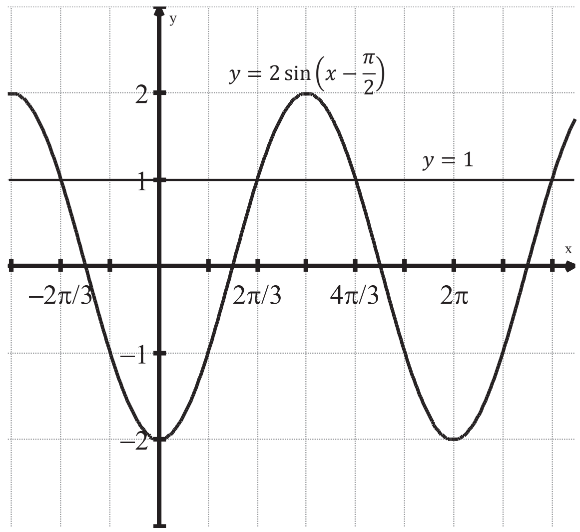
Discuss the effect a horizontal stretch has on the restricted domain. To solve the equation $\sin 3x = 1,\ 0 \le x < 2\pi$, for example, students find all possible solutions for $3x$ first. The restricted domain would be $0 \le 3x < 6\pi$. This results in $3x=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2}$ and $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2}$.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Using the graph shown, ask students to determine the general solution for $2\sin\left(x-\frac{\pi}{2}\right)-1=0$.



(T5.5)

- Ask students to determine how many solutions there are for the equation $3\sin\left[4\left(x-\frac{\pi}{2}\right)\right]=2$ on the interval $x \in [-\pi, 2\pi]$.

(T5.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

5.4 Equations and Graphs of Trigonometric Functions

SB: pp. 266-281

TR: pp. 134-138

Trigonometric Identities

Suggested Time: 15 Hours

Unit Overview

Focus and Context

In this unit, students will explore trigonometric relationships that always hold and others that hold only for given values of the angle. In this context, students will have the opportunity to derive proofs of trigonometric relationships and find algebraic solutions of trigonometric equations. Students will also simplify trigonometric expressions.

Outcomes Framework

GCO

Develop trigonometric reasoning.

SCO T6

Prove trigonometric identities, using:

- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent).

SCO T5

Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

Mathematical Processes

[C] Communication	**[PS]** Problem Solving
[CN] Connections	**[R]** Reasoning
[ME] Mental Mathematics and Estimation	**[T]** Technology
	[V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Trigonometry		
M4. Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. [C, CN, PS, R, T, V]	T1. Demonstrate an understanding of angles in standard position [0° to 360°]. [R, V] T2. Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V] T3. Solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]	T6. Prove trigonometric identities, using: • reciprocal identities • quotient identities • Pythagorean identities • sum or difference identities (restricted to sine, cosine and tangent) • double-angle identities (restricted to sine, cosine and tangent). [R, T, V] T5. Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

Trigonometry

Outcomes

Students will be expected to

T6 Prove trigonometric identities, using:

- **reciprocal identities**
- **quotient identities**
- **Pythagorean identities**
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent).

[R, T, V]

Elaborations—Strategies for Learning and Teaching

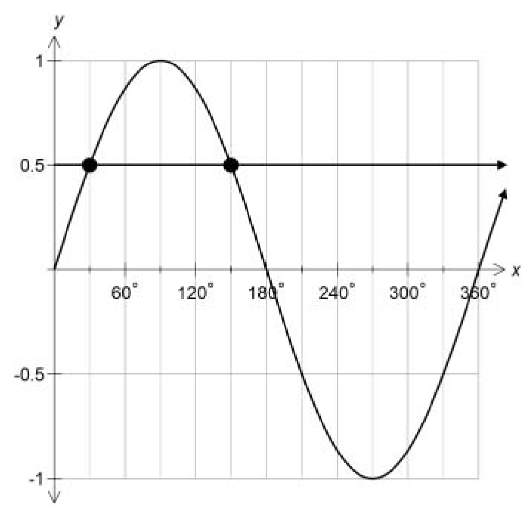
Students have had previous exposure to graphing trigonometric functions (T4) and solving trigonometric equations (T5). They were also introduced to the reciprocal trigonometric ratios (T3). Students are now introduced to trigonometric identities as trigonometric equations that are true for all permissible values of the variable in the expressions on both sides of the equation. They will verify identities both graphically and numerically, and prove identities using the Pythagorean identities, the quotient identities, the reciprocal identities, the sum/difference identities, and the double-angle identities.

Achievement Indicator:

T6.1 *Explain the difference between a trigonometric identity and a trigonometric equation.*

Students should be able to explain the difference between a trigonometric equation and a trigonometric identity. An identity is true for all permissible values, whereas an equation is only true for a smaller subset of the permissible values. This difference can be demonstrated with the aid of graphing technology.

Students have used the graphs of trigonometric functions to solve equations. The equation $\sin x = \frac{1}{2}$, for example, can be solved for $0^\circ \leq x < 360^\circ$, using the graphs of $y = \sin x$ and $y = \frac{1}{2}$:



The solutions to $\sin x = \frac{1}{2}$ for $0^\circ \leq x < 360^\circ$ are $x = 30^\circ$ and $x = 150^\circ$, which are the x-values of the intersection points.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Journal

- Ask students to respond to the following:

  Explain to a friend who missed today's class the difference between an identity and an equation.

(T6.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.1 Reciprocal, Quotient, and Pythagorean Identities

Student Book (SB): pp. 290-298

Teacher's Resource (TR): pp. 146-149

Trigonometry

Outcomes

Students will be expected to

T6 Continued ...

Achievement Indicators:

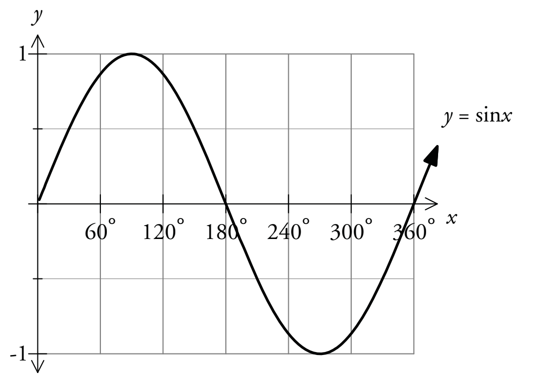
T6.1 *Continued*

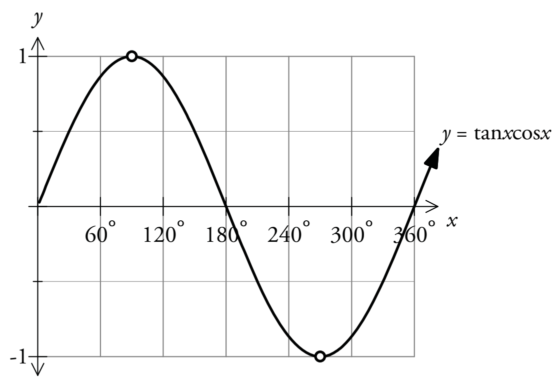
T6.2 *Determine, graphically, the potential validity of a trigonometric identity, using technology.*

T6.3 *Determine the non-permissible values of a trigonometric identity.*

Elaborations—Strategies for Learning and Teaching

Students could then work with the relation $\sin x = \tan x \cos x$. They should see that, when graphed, $y = \sin x$ and $y = \tan x \cos x$ are almost identical:





The only differences in the graphs occur at the points (90°, 1) and (270°, −1), which are non-permissible values of x. Therefore, $\sin x = \tan x \cos x$ is an identity since the expressions are equivalent for all permissible values.

Students should discuss why there are points for which identities are not equivalent. Non-permissible values for identities occur where one of the expressions is undefined. In the above example, $y = \tan x \cos x$ is not defined when $x = 90° + 180°n$, $n \in I$ since $y = \tan x$ is undefined at these values. Students should note that non-permissible values often occur when a trigonometric expression contains:

- a rational expression, resulting in values that give a denominator of zero
- tangent, cotangent, secant and cosecant, since these expressions all have non-permissible values in their domains.

Students should determine non-permissible values both graphically and algebraically.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine graphically if the following are identities. They should identify the non-permissible values.

  (i) $\sin\theta + \cos\theta\tan\theta = 2\sin\theta$

  (ii) $\tan^2\theta + 1 = \sec^2\theta$

  (iii) $\frac{\cos\theta}{\sin\theta} = \sec\theta$

(T6.1, T6.2, T6.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.1 Reciprocal, Quotient, and Pythagorean Identities

SB: pp. 290-298

TR: pp. 146-149

Trigonometry

Outcomes

Students will be expected to

T6 Continued ...

Achievement Indicators:

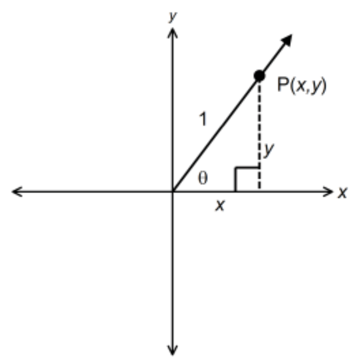
T6.4 *Verify a trigonometric identity numerically for a given value in either degrees or radians.*

T6.5 *Prove, algebraically, that a trigonometric identity is valid.*

Elaborations—Strategies for Learning and Teaching

Students should also verify numerically that an identity is valid by substituting numerical values into both sides of the equation. Angles in both degree and radian measures should be used.

To introduce the Pythagorean identities, students could be given the expression $\sin^2\theta + \cos^2\theta$ and asked to substitute in different values for θ. They should conclude inductively that $\sin^2\theta + \cos^2\theta = 1$ for all values of θ. Discuss how this approach is insufficient to conclude that the equation is an identity because only a limited number of values were substituted for θ, and there may be a certain group of numbers for which this identity does not hold. This discussion should lead to the idea of a proof – a deductive argument that is used to show the validity of a mathematical statement. To prove $\sin^2\theta + \cos^2\theta = 1$, the unit circle (T2), the definitions of sin θ and cos θ (T2), and the Pythagorean theorem can be used:



Encourage students to use the left side and right side notation where appropriate:

$$\begin{aligned} LS &= \sin^2\theta + \cos^2\theta \\ &= (y)^2 + (x)^2 \\ &= 1^2 \\ &= 1 \\ &= RS \end{aligned}$$

Using the above identity and the reciprocal and quotient identities, students should derive the other two Pythagorean identities ($1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$) and identify the non-permissible values. They should verify the identities numerically and validate them with proofs.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Journal

- Ask students to justify whether $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ is an identity, using numerical or graphical evidence.

(T6.1, T6.2, T6.3, T6.4)

Interview

- Ask students to explain whether or not $\sin \theta + \cos \theta = 1$ given that $\sin^2 \theta + \cos^2 \theta = 1$.

(T6.3, T6.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.1 Reciprocal, Quotient, and Pythagorean Identities

SB: pp. 290-298

TR: pp. 146-149

Trigonometry

Outcomes

Students will be expected to

T6 Continued ...

Achievement Indicator:

T6.6 *Simplify trigonometric expressions using trigonometric identities.*

Elaborations—Strategies for Learning and Teaching

Students should also simplify expressions using the Pythagorean identities, the reciprocal identities, and the quotient identities. Discuss specific strategies with students that they might use to begin the simplifications:

- Replace a "squared" term with a Pythagorean identity
- Write the expression in terms of sine or cosine
- For expressions involving addition or subtraction, it may be necessary to use a common denominator to simplify a fraction

Remind students to determine any non-permissible values of the variable in an expression. Students could be asked, for example, to identify the non-permissible values of θ in $\sin\theta\cos\theta\cot\theta$, and then simplify the expression.

Students often find simplifying trigonoemtric expressions more challenging than proving trigonometric identities because they may be uncertain of when an expression is simplified as much as possible. Developing a good foundation with simplifying expressions makes the transition to proving trigonometric identities easier.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to simplify $\frac{1-\sin^2\theta}{\cos\theta}$, identifying the non-permissible values.

(T6.3, T6.6)

Journal

- Ask students to respond to the following:

  For one of your homework problems, the answer is sec x csc x. You and your friend get different answers. Which of them is correct? Explain.

  (i) $\frac{\sec^2 x}{\tan x}$

  (ii) $\cot x + \tan x$

(T6.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.1 Reciprocal, Quotient, and Pythagorean Identities

SB: pp. 290-298

TR: pp. 146-149

Trigonometry

Outcomes

Students will be expected to

T6 Prove trigonometric identities, using:

- reciprocal identities
- quotient identities
- Pythagorean identities
- **sum or difference identities (restricted to sine, cosine and tangent)**
- **double-angle identities (restricted to sine, cosine and tangent).**

[R, T, V]

Achievement Indicators:

T6.7 *Determine, using the sum, difference and double-angle identities, the exact value of a trigonometric ratio.*

T6.2, T6.3, T6.4, T6.6 *Continued*

Elaborations—Strategies for Learning and Teaching

Frequently, trigonometric relationships involve angle measures that are related as either the sum or difference of other angles or the double of other angles. In such cases, students can use formulas to evaluate trigonometric functions. Once introduced to these, they should realize that the advantage in using the sum and difference formulas or the double-angle formulas is that resulting evaluations can be expressed as exact values rather than as approximate decimal values. The formulas are also used to simplify trigonometric expressions and verify identities.

Students should be exposed to the sum and difference identities for the primary trigonometric ratios:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

After working with the sum and difference identities, students should be exposed to the double-angle identities for the primary trigonometric ratios:

$$\sin 2a = 2\sin a \cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a = 2\cos^2 a - 1$$

$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$$

These identities should be verified numerically. Teachers could expose students to the derivation of these formulas but the proofs are not required for assessment.

Students should see that the three formulas for cos $2a$ are all equivalent and that they can use whichever one is most convenient in a problem.

A common student error is to express tan $2x$ as $2\tan x$ or $\cos 2x$ as $2\cos x$. Ensure students realize that these expressions are not equivalent.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the exact value of the following:
  (i) $\tan\left(\frac{23\pi}{12}\right)$
  (ii) $\sin(255°)$

(T6.7)

- If $\cos\theta = \frac{7}{25}$ where $270° \leq \theta < 360°$, ask students to determine the exact values of $\tan(2\theta)$ and $\sin\left(\theta + \frac{3\pi}{2}\right)$.

(T6.7)

Performance

- Ask students to explain whether or not $\sin 2x = 2\sin x$ is an identity.

(T6.7)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.2 Sum, Difference and Double-Angle Identities

SB: pp. 299-308

TR: pp. 150-153

Trigonometry

Outcomes

Students will be expected to

T6 Continued ...

Achievement Indicators:

T6.2, T6.3, T6.4, T6.5 T6.6, T6.7 *Continued*

Elaborations—Strategies for Learning and Teaching

The sum, difference, and double-angle identities are used to determine exact values of trigonometric expressions. Using the double-angle identities, students should determine the exact trigonometric ratios of angles that are not multiples of 30° or 45° but are multiples of 15°. They could algebraically determine, for example, the exact value of $\cos\frac{7\pi}{12}$ and $\tan 145°$. Students should also use these formulas to find the exact value of expressions such as $\frac{\tan 80° + \tan 55°}{1 - \tan 80° \tan 55°}$. They should realize that the sum and difference identities can be applied in either direction. They often ignore the fact that the sum and difference formulas are equally true when read from right to left.

The sum, difference, and double-angle identities are also needed to simplify certain trigonometric expressions. The expression $\frac{\sin 2x}{1-\cos 2x}$, for example, can be simplified to cot *x*, using the appropriate double-angle formula for cos 2*x*. Ask students to identify the non-permissible roots algebraically, and verify the solution numerically and/or graphically. To find the non-permissible roots algebraically, it is necessary to solve 1 - cos 2x ≠ 0. Students could:

- substitute the appropriate double-angle formula and solve:

$$\begin{aligned} 1 - \cos 2x &\neq 0 \\ 1 - \left(1 - 2\sin^2 x\right) &\neq 0 \\ 2\sin^2 x &\neq 0 \\ \sin^2 x &\neq 0 \\ \sin x &\neq 0 \\ x &\neq \pi n,\ n \in I \end{aligned}$$

- solve cos 2*x* ≠ 1, noticing that there is a horizontal stretch of $\frac{1}{2}$, which gives a period of π:

$$\begin{aligned} \cos 2x &\neq 1 \\ 2x &\neq 0 + 2\pi n,\ n \in I \\ x &\neq \pi n,\ n \in I \end{aligned}$$

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to use numerical examples to show that $\tan(x + y) \neq \tan x + \tan y$.

(T6.7)

Interview

- Students can work in pairs to prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$ in three ways, using a different identity for $\cos 2x$ in each proof. Ask them to discuss which identity they found the easiest to work with, and identify any strategies that they learned.

  The teacher then interviews the student pairs to gain insight into the level of understanding and ability to put mathematical ideas into words. Ask questions such as:

  (i) Which identity for $\cos 2x$ did you use first? Why did you choose this one?
  (ii) Which identity did you find easiest to work with?
  (iii) What strategies have you learned that might help you choose the best identity for $\cos 2x$ for future proofs?

  Rather than interview each student for each topic, teachers may decide to select a sample of students to interview, ensuring all students are included over time.

(T6.5)

Performance

- Prepare cards containing the steps in the proof of a trigonometric identity. Students work in small groups to decide on a logical sequence in which to place the cards. As students examine the cards, they should discuss their ideas about a possible sequence.

(T6.5)

Paper and Pencil

- Ask students to simplify:

  (i) $\frac{\cos 2x + \sin^2 x}{\sin 2x}$

  (ii) $\sin\left(\theta + \frac{\pi}{2}\right) - \sin\left(\theta - \frac{\pi}{2}\right)$

(T6.6, T6.7)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.2 Sum, Difference and Double-Angle Identities

SB: pp. 299-308

TR: pp. 150-153

Trigonometry

Outcomes

Students will be expected to

T6 Continued ...

Achievement Indicators:

T6.8 *Explain why verifying that the two sides of a trignometric identity are equal for given values is insufficient to conclude that the identity is valid.*

T6.3, T6.5 *Continued*

Elaborations—Strategies for Learning and Teaching

Students should use the reciprocal identities, the Pythagorean identities, the sum and difference identities, and the double-angle identities to prove other trigonometric identities. To prove identities, they must understand that one side of the identity is rewritten in terms of the functions found on the other side. Strategies for validating an identity include:

- writing expressions in terms of sine and cosine
- expressing the given trigonometric functions in terms of a single trigonometric function
- factoring expressions, including expressions with common factors, difference of squares, and trinomials
- writing expressions with a common denominator
- expanding an expression, such as multiplying two binomials together
- writing one fraction as two or more fractions
- multiplying by the conjugate
- multiplying an expression by a fraction equivalent to 1.

Students should also be reminded to identify any non-permissible roots when proving identities.

For the identity $(1 + \cot^2 x)(1 - \cos 2x) = 2$, the non-permissible roots are given by $\sin x \neq 0$ or $x \neq \pi n$, $n \in I$, since $\cot^2 x = \frac{\cos^2 x}{\sin^2 x}$. To prove this identity, encourage them to start with the side that appears more complicated. This proof uses both a Pythagorean identity and a double-angle identity. As they work through this proof, ask students how they decide which form of the double-angle formulas for cosine is most appropriate. They should see the importance of being able to express the trigonometric functions in terms of a single function.

Students can verify this numerically, but remind them that showing an identity is true for certain values of x is not a proof, since there may be other values of x that do not work in the identity. It does give evidence, however, that the identity is valid. They can also verify the result graphically.

Students should be exposed to proofs that require more than one strategy.

In some cases, both sides of an identity may be independently simplified to a common expression. Subsequently, the proof is validated by the transitive property.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to prove that the following identities are valid:

  (i) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

  (ii) $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan\theta$

  (iii) $1 + \sin 2\theta = (\sin\theta + \cos\theta)^2$

(T6.8)

- Ask students to graphically verify that $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$. They should determine the non-permissible roots and algebraically prove the identity.

(T6.3,T6.5,T6.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.3 Proving Identities

SB: pp. 309-315

TR: pp. 154-157

Trigonometry

Outcomes

Students will be expected to

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

[CN, PS, R, T, V]

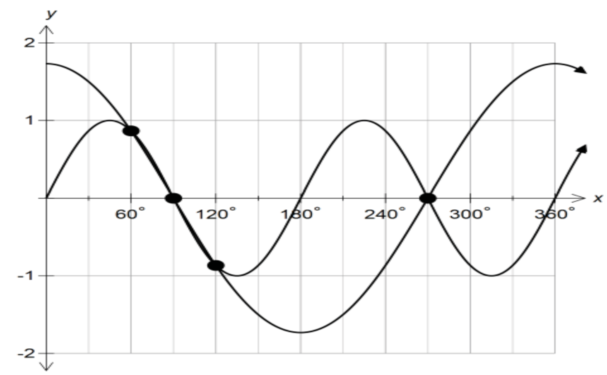
Achievement Indicator:

T5.1 *Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible.*

Elaborations—Strategies for Learning and Teaching

Students should recall solving trigonometric equations in Mathematics 2200 (T2) and earlier in this course (T5). The identities encountered earlier in this unit (T6) can now be applied to solve trigonometric equations.

Continue to emphasize the connection between graphical solutions and algebraic solutions for trigonometric equations. Students could, for example, be asked to find the solutions of $\sin 2x = \sqrt{3}\cos x$ for $0° \le x < 360°$. Use the graphs of $y = \sin 2x$ and $y = \sqrt{3}\cos x$ with domain $0° \le x < 360°$ to show the intersection at 60˚, 90˚, 120˚ and 270˚.



A double-angle identity is used to solve this same equation algebraically:

$$\sin 2x = \sqrt{3}\cos x$$
$$2\sin x\cos x - \sqrt{3}\cos x = 0$$
$$\cos x\left(2\sin x - \sqrt{3}\right) = 0$$
$$\cos x = 0 \qquad 2\sin x - \sqrt{3} = 0$$
$$x = 90°, 270° \qquad \sin x = \frac{\sqrt{3}}{2}$$
$$x = 60°, 120°$$

Students should see the relationship between the algebraic and graphical solutions.

Students should also solve trigonometric equations with unrestricted domains resulting in a general solution, in degrees and radian measure. Using the previous example with an unrestricted domain, the solutions in radians are $x = \frac{\pi}{2} + \pi n,\ x = \frac{\pi}{3} + 2\pi n,\ x = \frac{2\pi}{3} + 2\pi n,\ n \in I$. Here students see when the domain has no restriction there are an infinite number of solutions. Extending the graph so that students can see more intersection points may help consolidate understanding of general solutions.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve each of the following equations for $0° \le x \le 360°$, giving exact solutions where possible.

  (i) $\sin 2x = \sqrt{2}\cos x$

  (ii) $\tan 2x = \frac{2}{\sec^2 x}$

  Ask students to write the general solution for the above equations in degrees and radian measure.

  (T5.1, T5.2, T5.4)

- Ask students to solve $\cos 2x = \sin x$ graphically, and verify the answers numerically.

  (T5.4, T5.3)

Journal

- Ask students which identity should be used as a substitution for $\cos 2x$ when solving $1 - \cos 2x = \cos x$. Students should justify their answers and solve the equation with domain $0 \le \theta \le 2\pi$.

  (T5.1, T5.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.4 Solving Trigonometric Equations Using Identities

SB: pp. 316-321

TR: pp. 158-161

Trigonometry

Outcomes

Students will be expected to

T5 Continued ...

Achievement Indicators:

T5.3 *Verify, with or without technology, that a given value is a solution to a trigonometric equation.*

T5.2 *Determine, using technology, the approximate solution of a trigonometric equation.*

T5.4 *Identify and correct errors in a solution for a trigonometric equation.*

Elaborations—Strategies for Learning and Teaching

Students should verify that a particular value is a solution to a given trigonometric equation by simply substituting it into the equation and determining if it satisfies the equation.

Students should solve trigonometric equations for non-special angles as well. This can be done with the use of a scientific calculator where the degree or radian measure is found by finding the inverse trigonometric function of a ratio. Students could solve cos $2x + \sin^2 x = 0.7311$, for example, for the domain $0^\circ \leq x < 360^\circ$.

Students are also required to provide the general solution of a trigonometric equation. Remind them that the general solution of the equation above includes all angles that are coterminal with the solutions already found.

Identifying errors and providing the correct solution is a good technique for developing analytical skills. Students could be given a particular example with an error and asked to identify the mistake.

$$\sin^2 x - \sin x = 0$$
$$\sin^2 x = \sin x$$
$$\frac{\sin^2 x}{\sin x} = \frac{\sin x}{\sin x}$$
$$\sin x = 1$$
$$x = 90^\circ$$

In this case, a solution has been lost as a result of dividing both sides of the equation by sin x.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- A student's solution for $\tan^2 x = \sec x \tan^2 x$ for $0 \leq x < \pi$ is shown below:

$$\frac{\tan^2 x}{\tan^2 x} = \frac{\sec x \tan^2 x}{\tan^2 x}$$
$$1 = \sec x$$
$$x = 0,\ 2\pi$$

  Ask students to identify and explain the error(s).

  (T5.1, T5.4)

- Ask students to solve $\cos 2x = 0.8179$ for $0° \leq x \leq 360°$. They should also write the general solution in both degrees and radians.

  (T5.2)

Journal

- Distribute half sheets of paper or index cards and ask students to describe the "muddiest point" of solving trigonometric equations. They should jot down any ideas or parts of the lesson that were difficult to understand. This is a quick monitoring technique that allows any difficulties to be addressed.

  (T5.1, T5.2, T5.3, T5.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

6.4 Solving Trigonometric Equations Using Identities

SB: pp. 316-321

TR: pp. 158-161

Exponential Functions

Suggested Time: 12 Hours

Unit Overview

Focus and Context

In this unit, students are introduced to the graph of an exponential function $y = c^x$, $c > 0$, $c \neq 1$. They explore the effects of translations, stretches and reflections on the basic graph and apply these transformations to graph an exponential function of the form $y = a(c)^{b(t - h)} + k$.

Students solve exponential equations by writing both sides as rational powers of the same base. In cases where this is not possible, they use systematic trial or graphing technology to approximate a solution. This will be revisited in the next unit when they are introduced to logarithms.

Throughout the unit, students are exposed to problems that can be modelled using exponential functions.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF8

Graph and analyze exponential and logarithmic functions.

SCO RF9

Solve problems that involve exponential and logarithmic equations.

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions		
AN3. Demonstrate an understanding of powers with integral and rational exponents. [C, CN, PS, R] RF1. Interpret and explain the relationships among data, graphs and situations. [C, CN, R, T, V]	RF11. Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]	RF8. Graph and analyze exponential and logarithmic functions. [C, CN, T, V] RF9. Solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]

Relations and Functions

Outcomes

Students will be expected to

RF8 Graph and analyze exponential and logarithmic functions.

[C, CN, T, V]

Achievement Indicators:

RF8.1 *Sketch, with or without technology, a graph of an exponential function of the form $y = c^x$, $c > 0$, $c \neq 1$.*

RF8.2 *Identify the characteristics of the graph of an exponential function of the form $y = c^x$, $c > 0$, $c \neq 1$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students worked with the laws of exponents, integral exponents, and rational exponents. They also applied the exponent laws to expressions with rational and variable bases, as well as integral and rational exponents (AN3). Additionally, students graphed, with and without technology, a set of data and determined the restrictions on the domain and range (RF1). In Mathematics 2200, students graphed and analyzed reciprocal functions (limited to reciprocals of linear and quadratic functions) of the form $y = \frac{1}{f(x)}$, where they would have been introduced to the concept of asymptotes (RF11).

In the Function Transformations unit of Mathematics 3200, students worked with horizontal and vertical translations (RF1), horizontal and vertical stretches (RF2), and combinations of stretches and translations (RF3). Mapping rules were also studied in the context of these transformations. In this unit, they work with exponential functions.

As an introduction to exponential functions, students should create tables of values and their corresponding graphs for exponential functions of the form $y = c^x$ where c > 1 and $0 < c < 1$. They could create tables for functions such as the following:

- $y = 2^x$
- $y = \left(\frac{3}{2}\right)^x$
- $y = 2.8^x$
- $y = \left(\frac{2}{3}\right)^x$
- $y = 0.16^x$

When discussing features of these graphs, the focus should be on:

- *x*- and *y*-intercepts
- domain and range
- whether the graph is increasing or decreasing
- equation of the horizontal asymptote

As students think about the significance of the horizontal asymptote, they should consider in which cases the graph approaches its asymptote as *x*-values increase (tend toward positive infinity) and in which cases the graph approaches its asymptote as *x*-values decrease (tend toward negative infinity).

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- For each of the following, ask students to determine whether the function is increasing or decreasing and explain why.

  (i) $y = 4^x$

  (ii) $y = \left(\frac{1}{3}\right)^x$

  (iii) $y = \left(\frac{5}{2}\right)^x$

(RF8.2)

Journal

- Ask students to explain why an exponential function cannot have a negative base or a base that equals 0 or 1.

(RF8.2)

- Ask students to explain why graphs of functions of the form $y = c^x$, $c > 0$, $c \neq 1$:

  (i) do not have x-intercepts
  (ii) always have y-intercept (0, 1)
  (iii) always have the same domain and range

(RF8.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.1 Characteristics of Exponential Functions

Student Book (SB): pp. 334-345

Teacher's Resource (TR): pp. 174-180

Relations and Functions

Outcomes

Students will be expected to

RF8 Continued ...

Achievement Indicators:

RF8.1, RF8.2 *Continued*

Elaborations—Strategies for Learning and Teaching

Teachers should use this opportunity to define whether a function represents exponential growth (increases) or decay (decreases). Students should identify which functions are increasing or decreasing according to whether the base, c, is greater than 1 or between 0 and 1. They should also identify how the graphs are similar and how they are different.

Students should explore graphs of other exponential functions where $c>1$ and $0 < c < 1$ to confirm the conclusions reached in the introductory activity. They should also graph $y = c^x$ where $c = 1$ and where $c < 0$. When $c = 1$, a horizontal line is produced. When c is negative, if integer values of x are chosen the y-values "oscillate" between positive and negative values; for rational values of x, non-real values of y may be obtained. To see this, students could complete the table below for the functions $y = (-2)^x$ and $y = (1)^x$.

x	y
-2	
$-\frac{3}{2}$	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
$\frac{3}{2}$	
2	

Graphs and tables of values for exponential functions should also be analyzed by students to determine the corresponding functions.

Problems involving exponential growth and decay can be introduced here in contexts such as half-lives, bacterial growth/decay, light intensity, and finance, provided these situations involve functions of the form $y = c^x$. More extensive work involving functions of the form $y = a(c)^{b(x-h)} + k$ will be done later in the unit, once students have applied transformations to $y = c^x$. Students should identify any necessary restrictions on the domain and range due to the context of the problem.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students to determine which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). They should explain the choice.
  - (i) The number of neutrons present in a nuclear fission reaction triples at each stage of progression.
  - (ii) The volume of ice in a certain region of the arctic ice-cap is shrinking at a rate of 0.5% per year.
  - (iii) Lionel receives a pay increase of 2.5% per year.

(RF8.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.1 Characteristics of Exponential Functions

SB: pp. 334-345

TR: pp. 174-180

Relations and Functions

Outcomes

Students will be expected to

RF8 Continued...

Achievement Indicator:

RF8.3 *Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = c^x$, $c > 0$, $c \neq 1$ and state the characteristics of the graph.*

Elaborations—Strategies for Learning and Teaching

Previous work with transformations is now extended to include exponential functions. Students apply reflections, stretches and translations to exponential growth and decay curves, and then relate them to the parameters *a*, *b*, h, and *k* in a function of the form $y = a(c)^{b(x - h)} + k$, for a variety of values of *c.* Students could also write the mapping rules as they work with the transformations. It may be beneficial to first explore the types of transformations one at a time.

- Translations

Students could create tables of values and the graphs for functions of the form $f(x) = c^x + k$. Using functions such as $f(x) = 2^x$, $f(x) = 2^x + 3$, and $f(x) = 2^x - 4$, they should explore the connection between the value of *k* and the vertical translation. Students should then explore the effects of h for functions of the form $f(x) = c^{x - h}$, such as $f(x) = 2^{x + 3}$ and $f(x) = 2^{x - 4}$. Further exploration with other bases could be done with the aid of graphing technology. Students could also write the mapping rules relating the graph of $y = c^x$ to the transformed graphs.

- Stretches

Vertical stretches should be explored with functions of the form $y = a(c)^x$ by creating tables of values for a set of functions such as $y = 2(5)^x$, $y = 4(5)^x$ and $y = \frac{1}{3}(5)^x$. Horizontal stretches can be explored using a set of functions of the form $y = c^{bx}$ such as $y = 5^{2x}$, $y = 5^{\frac{x}{3}}$ and $y = 5^{0.2x}$.

- Reflections

Reflections across the *x*-axis should be explored for functions of the form $y = -c^x$ with tables of values for a set of functions such as $y = -2^x$, $y = -\left(\frac{1}{3}\right)^x$ and $y = -(4)^x$.

Reflections across the *y*-axis for functions of the form $y = c^{-x}$ can be explored by creating tables of values for a set of functions such as $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^{-x}$. Note that students should consider the relationship between $y = \left(\frac{1}{c}\right)^x$ and $y = c^{-x}$.

Once the effect of each transformation has been explored, students should work with combinations of transformations for functions of the form $y = a(c)^{b(x - h)} + k$, using a variety of values for the parameters *a*, *b*, h, and *k*. In the Function Transformations unit, students explored the order in which transformations should be applied. This concept should be reviewed here.

Applications of exponential growth or decay, such as cooling behaviour of a liquid, radioactive decay, medications, and light intensity, may be further addressed at this point.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to graph each of the following, giving the domain, range, and equation of the horizontal asymptote:

  (i) $y=-\frac{1}{3}(2)^{x+3}-1$

  (ii) $y=2(3)^{-\frac{1}{2}(x-2)}+5$

(RF8.3)

- Ask students to identify the value of each parameter (a, b, h, and k) and its effect on the original graph for the functions:

  (i) $y=-2\left(\frac{1}{2}\right)^{3(x-1)}+4$

  (ii) $y=\frac{1}{2}(3)^{-x+2}-4$

(RF8.3)

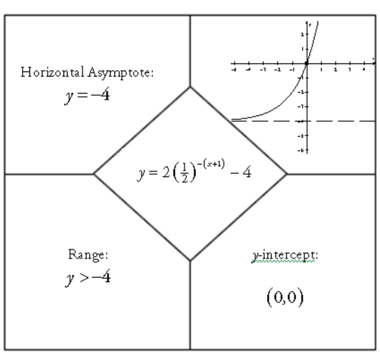
Journal

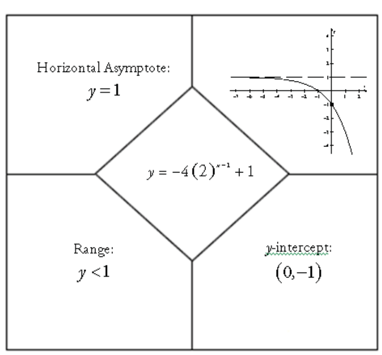
- Ask students to create an exponential decay function that has a horizontal asymptote at y = -4 and a y-intercept at (0, 2). They should create the graph of their function and explain why they selected the values of a, b, h, and k that they did.

(RF8.3)

Observation

- Students can work in pairs to complete puzzles containing the characteristics and graphs of various exponential functions of the form $y = a(c)^{b(x - h)} + k$. They should work with 20 puzzle pieces (4 complete puzzles) to correctly match the characteristics with each function. Sample puzzles are shown below.





(RF8.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.2 Transformations of Exponential Functions

SB: pp. 346-357

TR: pp. 181-186

Web Link

https://www.desmos.com/calculator

This online graphing calculator could be used to explore the effects of transformations on graphs of exponential functions.

Relations and Functions

Outcomes

Students will be expected to

RF9 Solve problems that involve exponential and logarithmic equations.

[C, CN, PS, R]

Achievement Indicators:

RF9.1 *Determine the solution of an exponential equation for which both sides can be written as rational powers of the same base.*

RF9.2 *Determine the solution of an exponential equation in which the bases are not rational powers of one another, using a variety of strategies.*

Elaborations—Strategies for Learning and Teaching

The focus here is on problem solving involving exponential equations using common bases and estimation methods. Problem solving situations using logarithmic equations will be explored in the next unit.

In Grade 9, students rewrote numbers as powers and solved problems involving the laws of exponents for whole number exponents (9N1, 9N2). In Mathematics 1201, they worked with negative exponents, rational exponents, the exponent laws, radicals, and problems involving laws of exponents and laws of radicals (AN3). They also worked with variable bases. Students now solve exponential equations with variable exponents where the bases can both be expressed as rational powers of the same base, including radical bases. They will work with equations such as $25^x = \left(\frac{1}{125}\right)^3$. In this case, both bases can be expressed as integer powers of 5.

Students should develop estimation skills for the solutions of exponential equations with variable exponents, including those where the bases cannot both be expressed as rational powers of the same base. When solving equations using logarithms, they will be better able to determine the reasonableness of solutions. Suggested strategies include systematic trial and graphing technology.

- Systematic Trial

Students could solve $2^x = 10$, correct to two decimal places, using the process outlined here:

Since 10 is closer to $2^3 = 8$ than to $2^4 = 16$, they might begin with $x \doteq 3.3$.

Test value for x	Power	Approximate value
3.3	$2^{3.3}$	9.849
3.4	$2^{3.4}$	10.556
The value obtained for $2^{3.3}$ is closer to 10, so the next estimate should be closer to 3.3 than to 3.4.		
3.31	$2^{3.31}$	9.918
3.32	$2^{3.32}$	9.987
3.33	$2^{3.33}$	10.056

They should reason that the best estimate is $x \doteq 3.32$ because 9.987 is closer to 10 than 10.056 is.

Students should use systematic trial to solve exponential equations with a variety of bases, including rational bases. Although the precision of estimates can vary, a minimum of one decimal place is required.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the solution for equations such as the following:
  (i) $9^{2x+1} = 81$
  (ii) $16^{2x+1} = \left(\frac{1}{2}\right)^{x-3}$
  (iii) $40 = 4^{2x+3}$

(RF9.1, RF9.2)

- Ask students to algebraically determine the solution for the following equations:
  (i) $\left(\frac{1}{3}\right)^{2x-1} = (81)^{3-x}$
  (ii) $5\left(\frac{1}{4}\right)^{x} = 80$
  (iii) $\sqrt{5} = 25^{x-1}$
  (iv) $27^{2x-1} = \sqrt[3]{3}$
  (v) $\sqrt[5]{8^{x-1}} = \sqrt[3]{16^{x}}$
  (vi) $\sqrt{3^{x}} = 9^{2x+1}$

(RF9.1)

Performance

- Using a *Quiz-Quiz-Trade* activity, students can solve a variety of exponential equations for which both sides can be written as rational powers of the same base.
  (i) Prepare a set of cards that at least matches the number of students in the class. One side of the card contains the question and the other side contains the answer.
  (ii) Give each student a card. Allow students a minute or two to become familiar with the question on the card. Then they find a partner to whom they will ask the question.
  (iii) If the partner is not able to answer the question, they should coach them first before providing the answer.
  (iv) After both partners are finished asking and answering the questions, they switch cards and find a new partner.

(RF9.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.3 Solving Exponential Equations

SB: pp. 358-365

TR: pp. 187-192

Notes

- Exponential equations with radical bases are not addressed in the student book.
- There is limited treatment in the student book of solving exponential equations where negative exponents are required.

Supplementing is necessary for these topics.

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicator:

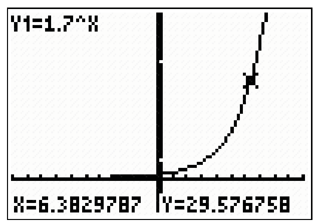
RF9.2 *Continued*

Elaborations—Strategies for Learning and Teaching

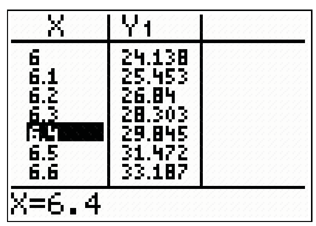
- Graphing Technology

Students could also use graphing technology to determine the solution for an exponential equation. With graphing technology, the solution can be found using the graph or a table of values.

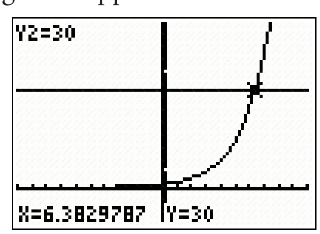
To determine the solution to the equation $1.7^x = 30$, for example, the graph of $y = 1.7^x$ can be used to determine the value of x that makes the value of the function approximately 30.



Students could also generate the table of values associated with the equation to find the value of x that makes the value of the function approximately 30.



Alternatively, they can determine the intersection of the graphs of $y = 1.7^x$ and $y = 30$ to get the approximate solution.



General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students if it is best to estimate the solution to $5^{3x-1} = 12^{2x+1}$ using systematic trial or graphing technology, and to defend their choice.

(RF9.2)

Journal

- Ask students to explain when they can solve an exponential equation algebraically and when they must use an estimation method (graphing or systematic trial).

(RF9.1, RF9.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.3 Solving Exponential Equations

SB: pp. 358-365

TR: pp. 187-192

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.3 *Solve a problem that involves exponential growth or decay.*

RF9.4 *Solve a problem that involves the application of exponential equations to loans, mortgages and investments.*

RF9.5 *Solve a problem by modeling a situation with an exponential or a logarithmic equation.*

Elaborations—Strategies for Learning and Teaching

Students should work with problems that can be modeled by exponential functions or equations, such as modeling the cooling behaviour of a liquid, radioactive decay, medications, half-lives, doubling time, bacterial growth/decay, light intensity, and finance. They should solve problems in situations where:

- an exponential function or equation is given
- the graph of an exponential function is given
- a situation is given and they have to create an exponential model to find the solution

Students should answer questions such as the following:

1. Shelly initially invests $500 and the value of the investment increases by 4% annually.
   - Create a function to model the situation.
   - How much money is in Shelly's investment after 30 years?
   - What amount of time will it take for the investment to double?

   Students should solve this problem using systematic trial, a table of values, or graphing technology.

2. The half-life of Radon 222 is 92 hours. From an initial sample of 48 g, how long would it take to decay to 6 g?

   Students could use either of the two methods below to solve this problem algebraically:

$A = 48\left(\frac{1}{2}\right)^n$	$A = 48\left(\frac{1}{2}\right)^{\frac{t}{92}}$
n is the number of 92 hour increments	t is the number of hours
$6 = 48\left(\frac{1}{2}\right)^n$	$6 = 48\left(\frac{1}{2}\right)^{\frac{t}{92}}$
$\frac{1}{8} = \left(\frac{1}{2}\right)^n$	$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{92}}$
$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^n$	$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{t}{92}}$
$3 = n$	$3 = \frac{t}{92}$
$\therefore$ It would take 276 hours.	$276 = t$

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- The half-life of a certain radioactive isotope is 30 hours. Ask students to algebraically determine the amount of time it takes for a sample of 1792 mg to decay to 56 mg.

(RF9.3)

- If a new car, purchased for $20 000, depreciates at a rate of 28% every two years, ask students to answer the following:
  (i) What will be the value of this car after 6 years?
  (ii) What amount of time will it take for the car to lose half its value?

(RF9.3, RF9.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.3 Solving Exponential Equations

SB: pp. 358-365

TR: pp. 187-192

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.3, RF9.4, RF9.5 *Continued*

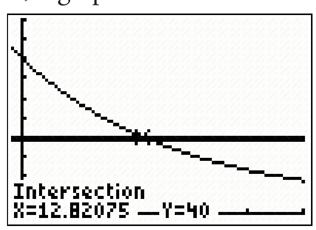
Elaborations—Strategies for Learning and Teaching

Students are also required to model a situation such as the following, using an exponential function, when given specific parameters.

A cup of hot chocolate is served at an initial temperature of 80°C and then allowed to cool in a stadium with an air temperature of 5°C. The difference between the hot chocolate temperature and the temperature of the room will decrease by 30% every 6 minutes. If T represents the temperature of the hot chocolate in degrees Celsius, measured as a function of time, *t*, in minutes, students can answer the following questions:

- What is the transformed exponential function in the form $T = a(c)^{b(t-h)} + k$? [Solution: $T = 75(0.7)^{\frac{t}{6}} + 5$]
- What is the temperature at time $t = 11$ minutes?
- How long does it take the hot chocolate to cool to a temperature of 40°C?

  As a possible method, a graphical solution is shown:



Students are required to model a situation with an exponential function from a given graph, a table of values, or a description when the equation of the horizontal asymptote is $y = 0$. When the asymptote is $y = 0$, students are expected to determine all parameters from the given information. When the asymptote is not $y = 0$, as in the above example, students are not required to determine the common ratio from a table of values or graph. They are required to determine the common ratio from the problem description.

To solve problems involving finance, students must become familiar with how annual interest rates are applied. Discuss common compounding periods. Students should understand that compounding semi-annually, for example, means interest is calculated twice a year.

Solving problems involving logarithmic equations will be addressed in the next unit.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

- Ask students to choose between two investment options and justify their choice: earning 12% interest per year compounded annually or 12% interest per year compounded monthly.

(RF9.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

7.3 Solving Exponential Equations

SB: pp. 358-365

TR: pp. 187-192

Logarithmic Functions

Suggested Time: 12 Hours

Unit Overview

Focus and Context

In the previous unit, students worked with exponential equations where both sides could be written as rational powers of the same base. They will now solve exponential equations where this is not possible.

Students are introduced to logarithms as inverses of exponential equations. They graph $y = \log_c x$, $c \neq 0$, $c > 1$ and explore the effects of various transformations on this graph.

Using the laws of logarithms, students simplify and evaluate expressions. They solve logarithmic equations and problems involving both exponential and logarithmic equations.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

- **SCO RF6**

  Demonstrate an understanding of logarithms.

- **SCO RF7**

  Demonstrate an understanding of the product, quotient and power laws of logarithms.

- **SCO RF8**

  Graph and analyze exponential and logarithmic functions.

- **SCO RF9**

  Solve problems that involve exponential and logarithmic equations.

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions		
RF1. Interpret and explain the relationships among data, graphs and situations. [C, CN, R, T, V] AN3. Demonstrate an understanding of powers with integral and rational exponents. [C, CN, PS, R]		RF6. Demonstrate an understanding of logarithms. [CN, ME, R] RF7. Demonstrate an understanding of the product, quotient and power laws of logarithms. [C, CN, ME, R, T] RF8. Graph and analyze exponential and logarithmic functions. [C, CN, T, V] RF9. Solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]

Relations and Functions

Outcomes

Students will be expected to

RF6 Demonstrate an understanding of logarithms.

[CN, ME, R]

Achievement Indicators:

RF6.1 *Explain the relationship between logarithms and exponents.*

RF6.2 *Express a logarithmic equation as an exponential equation and vice versa.*

RF6.3 *Determine, without technology, the exact value of a logarithm, such as* $\log_2 8$.

RF6.4 *Estimate the value of a logarithm, using benchmarks, and explain the reasoning; e.g., since* $\log_2 8 = 3$ *and* $\log_2 16 = 4$, $\log_2 9$ *is approximately equal to 3.1.*

Elaborations—Strategies for Learning and Teaching

In previous units in this course, students worked with inverses (RF5), and exponential functions and their graphs (RF8, RF9). They now build upon these concepts to study logarithms and their graphs.

Students should be introduced to logarithms as a different form of an exponential statement. The statement $3^2 = 9$, for example, can be written as $\log_3 9 = 2$. The base of the exponent is the same as the base of the logarithm. Students should understand that a logarithm $\log_c x = y$ is asking "What exponent, y, is needed so that $c^y = x$?". Given a statement in exponential form, they should be able to write it in logarithmic form, and vice versa. Introduce students to the common logarithm and note that, in this case, the base is usually not written: $\log_{10} x = \log x$.

When the value of a logarithm is a rational number, such as $\log_{64} 4$ and $\log_{\frac{1}{2}} 32$, students should be able to determine the exact value without technology. As well, students are required to determine an unknown value in logarithmic equations. The following examples can be solved by rewriting in exponential form:

- $\log_3 x = -2$
- $\log_x \frac{81}{16} = 2$

Students should also think about the restrictions on c for the equation $\log_c x = y$. Ask them to rewrite the following equations in exponential form to determine the value of x:

- $\log_1 4 = x$
- $\log_{-2} 8 = x$

From this they should see that when $c = 1$, the resulting exponential equation cannot be solved. The only exception is an equation with both $c = 1$ and $x = 1$. When $c < 0$, the resulting exponential equation may be unsolvable. Therefore, when working with logarithms, the base is restricted to positive values other than 1; that is, for $y = \log_c x$, $c > 0$, $c \neq 1$.

When solving an equation such as $\log_x \frac{81}{16} = 2$, students often forget that the base must be positive. Even though the quadratic equation $x^2 = \frac{81}{16}$ has two solutions ($x = \pm\frac{9}{4}$), the only solution for the logarithmic equation is $\frac{9}{4}$.

Students should also use benchmarks to estimate the value of a logarithm. To estimate $\log_5 106$, for example, they should notice that $5^2 = 25$ and $5^3 = 125$. Therefore, the answer must be between 2 and 3, and is closer to 3. Through systematic trial, students should be able to determine the value accurately to a minimum of one decimal place.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to evaluate the following without using technology:

  (i) $\log_{\frac{1}{5}} 25$

  (ii) $\log_4 32$

  (RF6.2,RF6.3)

- Ask students to determine the value of the following to one decimal place:

  (i) $\log_3 24$

  (ii) $\log_6 200$

  (iii) $\log_{\frac{1}{2}} 12$

  (RF6.4)

Interview

- Ask students to explain whether or not any positive real number can be the base of a logarithm.

  (RF6.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.1 Understanding Logarithms

Student Book (SB): pp. 372-382

Teacher's Resource (TR): pp. 200-204

Relations and Functions

Outcomes

Students will be expected to

RF8 Graph and analyze exponential and logarithmic functions.

[C, CN, T, V]

Achievement Indicator:

RF8.4 *Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.*

Elaborations—Strategies for Learning and Teaching

This outcome was addressed in the previous unit in relation to exponential functions of the form $y = c^x$, where $c > 0$, $c \neq 1$. In this unit, students are introduced to the graph of $y = \log_c x$, where $c > 0$, $c \neq 1$.

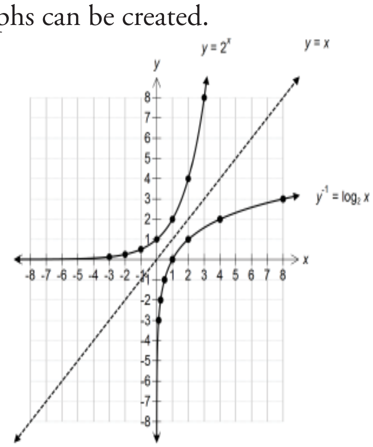
It is important that students understand the inverse relationship between exponential and logarithmic functions. They can use their previous knowledge of exponential functions and inverses to explore this relation. In the Function Transformations unit, students determined inverse equations by interchanging the x and y variables and solving for y (RF5). They can apply this procedure to determine the inverse of an exponential function, such as $f(x) = 2^x$. Writing this function as $y = 2^x$ and switching x and y leads to solving $x = 2^y$ for y. This requires writing the equation in logarithmic form and results in $y = \log_2 x$. The inverse function is $f^{-1}(x) = \log_2(x)$.

Students can use the table of values for $f(x) = 2^x$ to generate the table for $f^{-1}(x) = \log_2 x$ by interchanging the domain and range:

x	$f(x)=2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x	$f^{-1}(x)=\log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

From this, the graphs can be created.



Students should see that the graphs are reflections of each other in the line $y = x$. Ask them to identify the relationship between the domain and range for both functions. This method of graphing can be applied to any logarithmic function of the form $y = \log_c x$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- The point $\left(\frac{1}{36}, -2\right)$ is on the graph of $y = \log_c x$. If the point $(k, 216)$ is on the graph of the inverse, ask students to determine the values of c and k.

  (RF6.3, RF8.4)

- Given the graph and equation of an exponential function of the form $y = c^x$, ask students to determine the equation and the domain and range of the inverse relation.

  (RF8.4)

Journal

- Ask students to explain, using the graph of $y = \log_c x$, why they cannot evaluate $\log_c(-3)$and $\log_c(0)$.

  (RF8.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.1 Understanding Logarithms

SB: pp. 372-382

TR: pp. 200-204

Relations and Functions

Outcomes

Students will be expected to

RF8 Continued ...

Achievement Indicators:

RF8.5 *Sketch with or without technology, the graph of a logarithmic function of the form $y = \log_c x$, $c > 1$.*

RF8.6 *Identify the charactersitics of the graph of a logarithmic function of the form $y = \log_c x$, $c > 1$, including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.*

RF8.7 *Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, $c > 1$, and state the characteristics of the graph.*

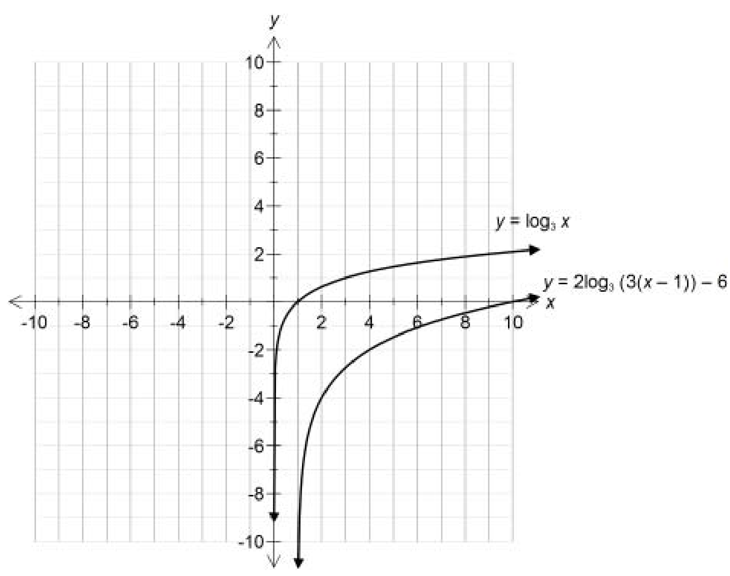
Elaborations—Strategies for Learning and Teaching

Students have been exposed to graphing using transformations for general functions (RF1, RF2, RF3, RF4) and for exponential functions (RF8). These transformations will now be applied to logarithmic functions. Logarithms will be restricted to bases that are greater than 1.

Students can now apply their previous work with transformations to logarithmic functions. Ask them to graph a logarithmic function, such as $y = \log_3 x$, and identify the domain, range, intercepts, and vertical asymptote. Students should notice that the domain is restricted by the vertical asymptote. Next, ask them to graph a transformation of this logarithmic function, such as $y = 2\log_3(3(x - 1)) - 6$. To graph this, students could identify the transformations:

- The vertical stretch is 2
- The horizontal stretch is $\frac{1}{3}$
- The horizontal translation is 1
- The vertical translation is -6

The graph can then be created by applying the stretches first, followed by the translations. Another approach is to create the mapping rule $(x, y) \rightarrow (\frac{1}{3}x + 1, 2y - 6)$ and then produce a table of values for the transformed function.



Students should recognize that the horizontal translation determines the vertical asymptote, which also defines the domain.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to graph each of the following using transformations:

  (i) $y = 2\log_2(-x+1) - 6$

  (ii) $y = -\frac{1}{2}\log_3(2(x+3)) + 1$

  (RF8.7)

- Ask students to identify the intercepts, vertical asymptote, and domain of $y = 2\log_6(3(x + 4)) - 4$.

  (RF8.7)

- Ask students to apply the mapping rule $(x, y) \rightarrow \left(-\frac{1}{2}x + 1, 3y - 12\right)$ to $y = \log_4 x$. They should write the resulting function and identify the domain, range, intercepts and vertical asymptote.

  (RF8.6)

Interview

- Ask students to explain why all functions of the form $y = \log_c x$ intersect at (1, 0).

  (RF8.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.2 Transformations of Logarithmic Functions

SB: pp. 383-391

TR: pp. 205-208

Relations and Functions

Outcomes

Students will be expected to

RF8 Continued ...

Achievement Indicators:

RF8.6, RF8.7 *Continued*

Elaborations—Strategies for Learning and Teaching

To determine the x-intercept of the transformed logarithmic function, students should solve the equation for $y = 0$. They are familiar with this from work with other types of functions. They can determine the x-intercept of $y = 2\log_3(3(x - 1)) - 6$, for example, as follows:

$$0 = 2\log_3(3(x - 1)) - 6$$

$$6 = 2\log_3(3x - 3)$$

$$3 = \log_3(3x - 3)$$

$$27 = 3x - 3$$

$$x = 10$$

Similarly, when determining the y-intercept, students let $x = 0$. At this point, there may be cases where it is necessary to use benchmarks to approximate the intercept. Later, they will be able to determine the y-intercepts more accurately and quickly with the use of a calculator. Students should also work with logarithmic functions that do not have a y-intercept. Evaluating $y = 2\log_3(3(x - 1)) - 6$ for $x = 0$, for example, results in $y = 2\log_3(-3) - 6$. Since the domain of $y = \log_3 x$ is $x \in (0, \infty)$, this cannot be evaluated.

Students may also have to factor an expression in order to determine the horizontal stretch. Before $y = 3\log_4(-5x - 5) - 4$ can be graphed, for example, it should be rewritten as $y = 3\log_4(-5(x + 1)) - 4$.

Students should graph logarithmic equations with a reflection in the y-axis to see the effect on the domain. They should recognize that for $y = 3\log_4(-5(x - 6)) + 1$, the vertical asymptote is $x = 6$, but the domain is $x \in (-\infty, 6)$ due to the reflection in the y-axis.

Students are not responsible for determining the equation of a logarithmic function given the graph. However, given the graph of a logarithmic function, they should identify the function from a list of options.

Students should sketch the graph of a given logarithmic function, clearly showing the asymptote and intercepts, and identify the graph of a given logarithmic function from a list of options.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Observation

- Students can work in pairs to complete puzzles containing the characteristics and graphs of various logarithmic functions. Refer to the observation strategy on page 147.

(RF8.6, RF8.7)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.2 Transformations of Logarithmic Functions

SB: pp. 383-391

TR: pp. 205-208

Note

Example 3 on p. 387 and #6 on p. 390 of the SB are not an expectation in this course.

Relations and Functions

Outcomes

Students will be expected to

RF7 Demonstrate an understanding of the product, quotient and power laws of logarithms.

[C, CN, ME, R, T]

Achievement Indicators:

RF7.1 *Develop and generalize the laws of logarithms, using numeric examples and exponent laws.*

RF7.2 *Derive each law of logarithms.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students worked with the laws of exponents, including integral and rational exponents. They also applied the exponent laws to expressions with rational and variable bases, as well as integral and rational exponents (AN3). The laws of logarithms will now be developed using both numerical examples and the exponent laws.

Students will work with the following laws of logarithms, with the conditions $c > 0$, $m > 0$, $n > 0$ and $c \neq 1$ where $c, m, n \in R$:

- Product Law: $\log_c MN = \log_c M + \log_c N$
- Quotient Law: $\log_c \frac{M}{N} = \log_c M - \log_c N$
- Power Law: $\log_c M^P = P\log_c M$

The focus here is to use several numerical examples to allow students to develop the laws of logarithms.

To develop the quotient law, for example, a procedure similar to the following could be used:

- Begin with the exponential equations: $16 = 4^2$ and $64 = 4^3$
- Convert each equation to logarithmic form:
  $\log_4 16 = 2$ and $\log_4 64 = 3$
- Divide the equations from Step 1: $\frac{64}{16} = 4^{3-2}$
- Rewrite using logarithmic form: $\log_4 \frac{64}{16} = 3 - 2$
- Substitute $\log_4 16$ in for 2 and $\log_4 64$ in for 3:
  $\log_4 \frac{64}{16} = \log_4 64 - \log_4 16$

Students could evaluate each side of the equation to verify that the left hand side of the equation is indeed the same as the right hand side. They should then derive the general case for the quotient law. A similar exercise could be used to develop the product law.

An example such as the following could be used for the power law:

- Start with the exponential equation: $16 = 4^2$ (i.e., $\log_4 16 = 2$)
- Raise both sides of the equation to the power p: $16^p = (4^2)^p$
- Simplify the equation: $16^p = (4)^{2p}$
- Convert the equation to logarithmic form: $\log_4 16^p = 2p$
- Substitute $\log_4 16$ in for 2: $\log_4 16^p = (\log_4 16)(p)$
- Rewrite: $\log_4 16^p = p\log_4 16$

From this, students should derive the general case for the power law. Students should be exposed to the proofs of the laws of logarithms, but not required to reproduce these proofs.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to verify using the logarithm laws:

  (i) $\log_3 27 = \log_3 9 + \log_3 3$
  (ii) $\log_5 25 = \log_5 125 - \log_5 5$
  (iii) $\log_2 64 = 6 \log_2 2$

(RF7.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.3 Laws of Logarithms

SB: pp. 392-403

TR: pp. 209-213

Relations and Functions

Outcomes

Students will be expected to

RF7 Continued ...

Achievement Indicators:

RF7.3 *Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.*

RF7.4 *Determine, with technology, the approximate value of a logarithmic expression, such as* $\log_2 9$.

Elaborations—Strategies for Learning and Teaching

Students are expected to simplify logarithmic expressions involving both numerical and variable arguments. For expressions with variable arguments, they should determine the restrictions on the variable(s). The arguments should be restricted to polynomials of degree 2 or less. They were introduced to factoring techniques in Mathematics 1201 (AN5). They should now be exposed to problems where factoring is necessary in order to simplify the expression. Students should also be exposed to questions where they are required to write a logarithmic expression as a single logarithm and simplify if necessary:

- $\log_6 4 - (\log_6 72 + \frac{1}{4}\log_6 16)$
- $\log_b 2x + 3(\log_b x - \log_b y)$

Conversely, they should write a single logarithm as the sum and difference of multiple logarithms. Applying the laws of logarithms to expand or condense logarithmic expressions is useful in solving logarithmic equations.

Students should note that there is no general property of logarithms that can be used to simplify $\log_c(x + y)$. They sometimes mistakenly think that this expression is equal to $\log_c x + \log_c y$. To verify that this is not true, students could evaluate logarithmic expressions such as:

- $\log_{10}(2 + 3) = \log_{10} 5 \approx 0.699$
- $\log_{10} 2 + \log_{10} 3 \approx 0.301 + 0.477 = 0.778$

Students were introduced to evaluating logarithmic expressions without the use of technology by using benchmarks (RF6). They will now extend this to approximating the solution, using technology. To determine the approximate value of $\log_2 9$, for example, the equation $\log_2 9 = x$ can be rewritten in exponential form:

$$2^x = 9$$
$$\log 2^x = \log 9$$
$$x \log 2 = \log 9$$
$$x = \frac{\log 9}{\log 2}$$
$$x \approx 3.17$$

As students work through this example, they should be able to connect the solution of the problem to the original equation.

$$\log_2 9 = x \rightarrow x = \frac{\log 9}{\log 2}$$

This leads to the property $\log_a b = \frac{\log b}{\log a}$. The logarithmic expression can be evaluated using a calculator, as the base has been changed to 10.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:

  (i) A student was asked to simplify $\log_2 16 - \log_2 32 + 2\log_2 4$ and provided the solution:

  $$\log_2 16 - \log_2 32 + \log_2 8$$
  $$\log_2 \tfrac{16}{32} + \log_2 8$$
  $$\log_2 \tfrac{1}{2} + \log_2 8$$
  $$\log_2 4$$
  $$2$$

  State where the error first occurred and write the correct solution to the problem.

  (RF7.3)

  (ii) If $P = \log_3 8$ and $Q = \log_3 6$, write $\log_3 8\sqrt{6}$ in terms of P and Q.

  (RF7.3)

- Ask students to evaluate using the laws of logarithms:

  (i) $3\log_6(2) + \log_6(27)$

  (ii) $\log_5(2.5) + 2\log_5(10) - \log_5(2)$

  (RF7.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.3 Laws of Logarithms

SB: pp. 392-403

TR: pp. 209-213

Note

The property $\log_a b = \frac{\log b}{\log a}$ is not developed in the SB, but is referenced in question #19 on p. 402.

Relations and Functions

Outcomes

Students will be expected to

RF9 Solve problems that involve exponential and logarithmic equations.

[C, CN, PS, R]

Achievement Indicators:

RF9.6 *Determine the solution of a logarithmic equation, and verify the solution.*

RF9.7 *Explain why a value obtained in solving a logarithmic equation may be extraneous.*

RF9.2 *Determine the solution of an exponential equation in which the bases are not rational powers of one another, using a variety of strategies.*

Elaborations—Strategies for Learning and Teaching

In the previous unit, students solved exponential equations where the powers could both be expressed as rational powers of the same base. In cases where the bases were not the same graphing technology and systematic trial were used to estimate the value of the variable (RF9). Students will now use logarithms to solve these equations.

Ask students to solve logarithmic equations that require them to find solutions for both linear and quadratic equations:

- $\log_5(x + 1) + \log_5(x - 2) = \log_5 4$
- $\log_2(4x - 1) - \log_2(2x + 1) = 3$

When solving equations, non-permissible values of the variable must be considered. Solving $\log_2(x + 6) + \log_2(x + 4) = \log_2 8$, for example, results in the roots $x = -2$ and $x = -8$. Students should note that $x = -2$ is permissible, while $x = -8$ is extraneous.

Students should now use logarithms to solve exponential equations. They should be able to give answers as both approximate and exact values. When solving $3^{x+1} = 5^{4x-3}$ students can state the answer as $x = \frac{-3\log 5 - \log 3}{\log 3 - 4\log 5}$ or $x \approx 1.11$.

A common error that students may make is not using the distributive property correctly when multiplying the logarithm and the variable expression. Encourage them to put brackets around the exponent portion of the equation when moving it to the front of the logarithm.

They will also have to verify that the solution is not extraneous. Students should be encouraged to use graphing technology or systematic trial to check the reasonableness of their solution. Verification can also be done by substituting the solution back into the original equation.

A common student error occurs when students solve questions similar to $7(2^{3x}) = 21$. Discuss with students why $7(2^{3x})$ cannot be written as 14^{3x}. Students should divide both sides of the equation by 7 in order to isolate the power ($2^{3x} = 3$).

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

- Ask students to create a logarithmic equation where the only solution is $x = 4$.

(RF9.6)

- Students should explain why the equation $\log_4(1 - x) + \log_4(x - 3) = 1$ has no roots.

(RF9.6, RF9.7)

Interview

- Discuss with the student why the solution to $\log_4(x + 2) + \log_4(x - 4) = 2$ must be in the domain $\{x > 4, x \in R\}$.

(RF9.7)

Paper and Pencil

- Ask students to identify the error in the given solution and explain why it is incorrect. They should then write the correct solution.

$$10^x + 5 = 60$$
$$\log\left(10^x + 5\right) = \log 60$$
$$\log 10^x + \log 5 = \log 60$$
$$x \log 10 = \log 60 - \log 5$$
$$x = \frac{\log 60 - \log 5}{\log 10}$$

(RF9.2)

Performance

- Students could work in small groups to put together a jigsaw puzzle where the expressions on the adjacent sides of the puzzle pieces have to be equivalent. This activity provides students with the opportunity to practice work with logarithms, make mathematical arguments about whether or not pieces fit together, and check and revise their work.

(RF7.3, RF7.4, RF9.2, RF9.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.4 Logarithmic and Exponential Equations

SB: pp. 404-415

TR: pp. 214-218

Web Link

www.mmlsoft.com/index.php?option=com_content&task=view&id=11&Itemid=12

Tarsia is a software program for creating jigsaw puzzles.

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.3 *Solve a problem that involves exponential growth or decay.*

RF9.5 *Solve a problem by modeling a situation with an exponential or a logarithmic equation.*

RF9.8 *Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.*

Elaborations—Strategies for Learning and Teaching

In the previous unit, students solved problems involving exponential growth and decay, as well as financial applications (RF9). These types of problems should now be revisited, as the majority of them involve exponential equations that are solved using logarithms.

Students should solve problems where:

- the logarithmic or exponential equation is given
- the graph of an exponential function is given
- a situation is given that can be modeled by an exponential or logarithmic equation.

Students will solve problems involving logarithmic scales such as the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution), and decibel scale (used to measure sound level). When dealing with Richter scale, pH scale or decibel scale problems, students are not expected to develop formulas but should be given the formula when it is required.

Students may be familiar with pH scales from work in science courses. The pH scale of a solution is determined using the equation $y = -\log x$, where x is the concentration of hydrogen ions in moles per litre (mol/L). The scale ranges from 0 to 14 with the lower numbers being acidic and the higher numbers being basic. A value of pH = 7 is considered neutral. The scale is a logarithmic scale with one unit of increase in pH resulting in a 10 fold decrease in acidity. Another way to look at this would be that a one unit increase in pH results in a 10 fold increase in basicity.

The magnitude of an earthquake, y, can be determined using $y = \log x$, where x is the amplitude of the vibrations measured using a seismograph. An increase in one unit in magnitude results in a 10 fold increase in the amplitude.

Sound levels are measured in decibels using $\beta = 10(\log I + 12)$, where β is the sound level in decibels (dB) and I is the sound intensity measured in watts per metre squared (w/m^2). This would be a good opportunity for students to measure audio volumes in the environment around them. They could use a smartphone application, for example, to show the approximate decibel level of their location. Although quite accurate, the application is mainly a tool for detecting noise level in casual settings.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:

  (i) The magnitude of an earthquake, M, as measured on the Richter scale is given by M = log I, where I is the intensity of the earthquake (measured in micrometres from the maximum amplitude of the wave produced on a seismograph). A town experiences an earthquake with a magnitude of 4.2 on the Richter scale. Four years later, the same town experiences an earthquake that is 5 times as intense as the first earthquake. What is the magnitude of the second earthquake?

  (RF9.8)

  (ii) After taking a cough suppressant, the amount, A, in mg, remaining in the body is given by $A = 10(0.85)^t$, where t is given in hours.
  (a) What was the initial amount taken?
  (b) What percent of the drug leaves the body each hour?
  (c) How much of the drug is left in the body 6 hours after the dose is administered?
  (d) How long is it until only 1 mg of the drug remains in the body?

  (RF9.3)

- Ask students to answer the following questions using the table below:

Location and Date	Magnitude
Chernobyl - 1987	4
Haiti - January 12, 2012	7
Northern Italy - May 20, 2012	6

(i) How many times as intense was the earthquake in Haiti compared to the one in Chernobyl?
(ii) How many times as intense was the earthquake in Haiti compared to the one in Northern Italy?
(iii) How many times as intense was the earthquake in Northern Italy compared to the one in Chernobyl?
(iv) If a recent earthquake was half as intense as the one in Haiti, what would be the approximate magnitude?

(RF9.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.4 Logarithmic and Exponential Equations

SB: pp. 404-415

TR: pp. 214-218

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.4 *Solve a problem that involves the application of exponential equations to loans, mortgages and investments.*

RF9.5 *Continued*

Elaborations—Strategies for Learning and Teaching

Questions involving finance should also be revisited. In the previous unit, students were exposed to situations where the bases could be written the same. They will now work with questions where logarithmic equations are used. A reminder of the different compounding periods may be necessary.

The formula $A = A_0(1 + r)^n$ can be used for finance calculations, where A_0 is the initial value, r is the interest rate per compounding period and n is the number of compounding periods. For example, if a \$1000 deposit is made at a bank that pays 12% interest compounded monthly, students should be able to determine, using logarithms, how long it will take for the investment to reach \$2000. Students should also be exposed to situations where it is necessary to determine the initial value, the interest rate, or the number of compounding periods.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Journal

- Ask students to create a flowchart of the steps needed to solve exponential equations and a flowchart of the steps needed to solve logarithmic equations. Then have them compare the flowcharts for similarities. Ask them to consider if one flowchart can be designed that handles both types of equations.

(RF9)

Performance

- Students can work in groups of two for the activity *Pass the Problem*. Each pair gets a problem that involves a situation to be modelled with an exponential or a logarithmic equation. Ask one student to write the first line of the solution and then pass it to the second student. The second student verifies the workings and checks for errors. If there is an error, students should discuss what the error is and why it occurred. The student then writes the second line of the solution and passes it to their partner. This process continues until the solution is complete.

(RF9.3, RF9.4, RF9.5, RF9.8)

Resources/Notes

Authorized Resource

Pre-Calculus 12

8.4 Logarithmic and Exponential Equations

SB: pp. 404-415

TR: pp. 214-218

Permutations, Combinations and The Binomial Theorem

Suggested Time: 14 Hours

Unit Overview

Focus and Context

In this unit, students are introduced to the fundamental counting principle and the concept of counting with permutations, where the arrangement of elements in the event is important. They are also introduced to the role and significance of $n!$. The ability to rewrite a factorial in several different ways can be helpful in simplifying expressions with binomial coefficients.

Students also see that the process of counting events where the arrangement of elements is not important requires a different approach. For this, they are introduced to counting with combinations.

Finally, they are introduced to the binomial theorem and how it is used to expand a binomial that is raised to a power. They see that Pascal's triangle is a convenient pattern for writing binomial coefficients.

Outcomes Framework

GCO

Develop algebraic and numeric reasoning that involves combinatorics.

SCO PCBT1

Apply the fundamental counting principle to solve problems.

SCO PCBT2

Determine the number of permutations of n elements taken r at a time to solve problems.

SCO PCBT3

Determine the number of combinations of n different elements taken r at a time to solve problems.

SCO PCBT4

Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Permutations, Combinations and Binomial Theorem		
AN4. Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. [CN, R, V]	AN5. Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, ME, R]	PCBT1. Apply the fundamental counting principle to solve problems. [C, PS, R, V] PCBT2. Determine the number of permutations of *n* elements taken *r* at a time to solve problems. [C, PS, R, V] PCBT3. Determine the number of combinations of *n* different elements taken *r* at a time to solve problems. [C, PS, R, V] PCBT4. Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT1 Apply the fundamental counting principle to solve problems.

[C, PS, R, V]

Achievement Indicator:

PCBT1.1 *Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.*

Elaborations—Strategies for Learning and Teaching

The fundamental counting principle is a means of finding the number of ways of performing two or more operations together. It will be developed by solving problems through the use of graphic organizers such as lists and tree diagrams. Later, the fundamental counting principle will be applied in work with permutations and combinations.

Tree diagrams were used in Grade 7 (7SP5 and 7SP6) and Grade 8 (8SP2) to determine the number of possible outcomes in probability problems. Students now apply tree diagrams and other graphic organizers to counting problems. The following example could be used to activate students' prior knowledge.:

The school cafeteria advertises that it can serve up to 24 different meals consisting of one item from each of three categories:

Fruit: Apples, Bananas, or Cantaloupe

Sandwiches: Roast Beef or Turkey

Beverages: Lemonade, Milk, Orange Juice or Pineapple Juice

Is their advertising accurate?

	Fruits	Sandwiches	Beverages
Meals with Apples	A	R	L
	A	R	M
	A	R	O
	A	R	P
	A	T	L
	A	T	M
	A	T	O
	A	T	P
Meals with Bananas	B	R	L
	B	R	M
	B	R	O
	B	R	P
	B	T	L
	B	T	M
	B	T	O
	B	T	P
Meals with Cantaloupe	C	R	L
	C	R	M
	C	R	O
	C	R	P
	C	T	L
	C	T	M
	C	T	O
	C	T	P

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Jill is trying to select a new cell phone based on the following:

  Brands: Ace, Best, Cutest

  Colour: Lime, Magenta, Navy, Orange

  Plans: Text, Unlimited Calling

  Ask students to construct a tree diagram, a table, and an organized list to determine the number of ways of selecting Jill's new cell phone.

(PCBT1.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

Student Book (SB): pp. 516-527

Teacher's Resource (TR): pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

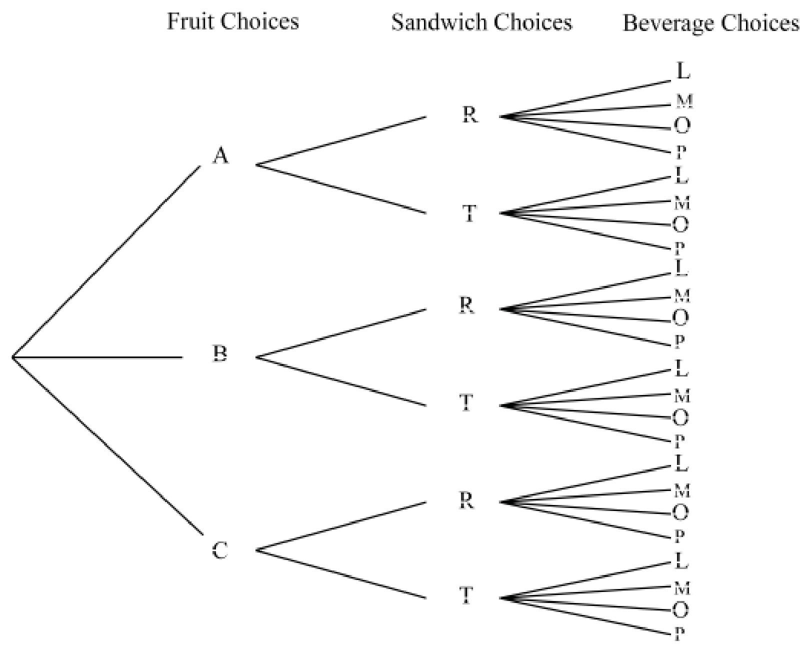
Students will be expected to

PCBT1 Continued ...

Achievement Indicator:

PCBT1.1 *Continued*

Elaborations—Strategies for Learning and Teaching



Students should be reminded that the possible meals indicated by the table and tree diagram could also be written as an organized list.

Meals with Apples	Meals with Bananas	Meals with Cantaloupe
ARL	BRL	CRL
ARM	BRM	CRM
ARO	BRO	CRO
ARP	BRP	CRP
.	.	.
.	.	.
.	.	.

Modifying this example to include a fourth category or adding more options in one of the categories can be used to illustrate limitations on the practicality of using graphic organizers for counting problems and offers a good introduction to the fundamental counting principle. Sometimes the task of listing and counting all the outcomes in a given situation is unrealistic, since the sample space may be very large. The fundamental counting principle enables students to find the number of outcomes without listing and counting each one.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT1 Continued ...

Achievement Indicators:

PCBT1.2 *Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.*

PCBT1.3 *Solve a simple counting problem by applying the fundamental counting principle.*

Elaborations—Strategies for Learning and Teaching

In the previous example, there are three fruit options. For each of these, there are two sandwich choices, and for each sandwich choice there are four beverage choices. Multiplying the number of options from each category gives possible meal choices, which agrees with the previous result. This illustrates the fundamental counting principle, which states that if there are a ways to perform a task, b ways to perform a second task, c ways to perform a third task, etc., then the number of ways of performing all the tasks together is $a \times b \times c \times \ldots$

Point out to students that when choosing a fruit and a sandwich and a beverage, the word "and" indicates these three selections (operations) are performed together, so the number of ways of doing each individual selection are multiplied. If, instead, a fruit or a sandwich or a beverage is being selected, then the possibilities are:

3 fruit choices + 2 sandwich choices + 4 beverage choices

= 9 possibilities.

In effect, the problem would become the same as if there were only 9 items on the menu and only one of those items was being selected. Ask students how many different selections would be possible in this case.

The intent of this discussion is instructional in nature and is meant to help students understand why, when using the fundamental counting principle, the individual choices are multiplied rather than added. It is not intended that students be explicitly evaluated on the use of "and" versus "or".

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Journal

- Ask students to respond to the following:

  Your school cafeteria offers three salads, four main courses, two vegetables, and three desserts.

  With the aid of a tree diagram, a table, and/or an organized list, explain why it makes sense to multiply the options from each category to determine the total number of possible meals if a salad, a main course, a vegetable and a dessert is included in the meal.

  (PCBT1.2)

Paper and Pencil

- Ask students to answer the following:

  (i) Sheldon has a red, a green and a blue shirt. He also has a pair of brown pants, a pair of beige pants, and a pair of black pants. His sock drawer contains one pair of black socks and one pair of grey socks. His shoe rack has one pair of moccasins, one pair of loafers, a pair of rubber boots, and a pair of sneakers. Determine the number of ways Sheldon can select an outfit consisting of one item from each category.

  (ii) In Newfoundland and Labrador, a license plate consists of a letter-letter-letter-digit-digit-digit arrangement such as CXT 132.
  (a) How many license plates are possible?
  (b) How many license plates are possible if no letter or digit can be repeated?
  (c) How many license plates are possible if vowels (a, e, i, o, u) are not allowed?

  (iii) Canadian postal codes consist of a letter-digit-letter-digit-letter-digit arrangement.
  (a) How many codes are possible, and how does this compare with the number of license plates in Newfoundland and Labrador?
  (b) In Newfoundland and Labrador, all postal codes begin with the letter A. How many postal codes are possible?

  (PCBT1.3)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT2 Determine the number of permutations of *n* elements taken *r* at a time to solve problems.

[C, PS, R, V]

Achievement Indicators:

PCBT2.1 *Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.*

PCBT2.2 *Determine, in factorial notation, the number of permutations of n different elements taken n at a time to solve a problem.*

Elaborations—Strategies for Learning and Teaching

The concept of a permutation will be explored through the use of graphic organizers. A formula will be developed and applied in problem solving situations, including those that involve permutations with constraints.

A permutation is an ordered arrangement of all or part of a set. For example, the possible permutations of the letters A, B and C are ABC, ACB, BAC, BCA, CAB and CBA. The order of the letters matters.

Students should first be introduced to permutations of *n* different elements taken *n* at a time and will then move to permutations of *n* different elements taken *r* at a time.

Students should learn to recognize and use *n*! to represent the number of ways to arrange *n* distinct objects. For example, how many ways are there to arrange or permute a group of five people in a line?

1st Person	2nd Person	3rd Person	4th Person	5th Person
5 options	4 options remaining	3 options remaining	2 options remaining	1 option remaining

By the fundamental counting principle, there are 5 × 4 × 3 × 2 × 1 or 120 ways. This product can be written in compact form as 5! Generally, $n! = n(n-1)(n-2)(n-3)\ldots(2)(1)$ where $n \in N$.

Note that 0! = 1 will be addressed in the context of permutations after the formula for ${}_nP_r$ is introduced.

In preparation for working with formulas for permutations and combinations, students should simplify factorial expressions such as:

- $\dfrac{100!}{97!} = \dfrac{100 \times 99 \times 98 \times 97!}{97!} = 100 \times 99 \times 98$
- $\dfrac{n!}{(n-2)!} = \dfrac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = n^2 - n$
- $\dfrac{3!(n+1)!}{2!(n-2)!} = \dfrac{3 \times 2!(n+1)n(n-1)(n-2)!}{2!(n-2)!}$

  $= 3(n+1)n(n-1)$

  $= 3n^3 - 3n$

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Performance

- The code for a lock consists of three numbers selected from 0, 1, 2, 3 with no repeats. For example, the code 1-2-1 would not be allowed but 3-0-2 would be allowed. Ask students to use a tree diagram or other graphic organizer to determine the number of possible codes and explain why this is a permutation problem.

(PCBT2.1)

Journal

- Ask students to explain why a typical so-called "combination lock" used on school lockers should more properly be called a permutation lock.

(PCBT2.1)

Paper and Pencil

- Ask students to answer the following:

(i) In how many different ways can a set of 5 distinct books be arranged on a shelf?

(ii) In how many different orders can 15 different people stand in a line?

(iii) Simplify:

(a) $\dfrac{1000!}{998!}$

(b) $\dfrac{1000!}{998!1000}$

(c) $\dfrac{4!(n+1)!}{3!(n-1)!}$

(d) $\dfrac{(n+2)!}{(n+4)!}$

(iv) Determine the number of ways of selecting a president, a secretary, and a treasurer from a group of 10 people if no person can hold more than one position.

(PCBT2.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT2 Continued ...

Achievement Indicators:

PCBT2.3 *Determine, using a variety of strategies, the number of permutations of n different elements taken r at a time to solve a problem.*

PCBT2.4 *Explain why n must be greater than or equal to r in the notation* ${}_nP_r$.

Elaborations—Strategies for Learning and Teaching

Solving a simple counting problem can help students develop a formula for determining the number of permutations of *n* different elements taken *r* at a time, which will be more efficient when working with larger values of *r*. Ask students to determine how many ways there are to arrange any three of a group of five people in a line. They should reason that there are 5 options for the first position, 4 options remaining for the second position, and 3 options remaining for the third position. By the fundamental counting principle, there are 5 × 4 × 3 = 60 permutations. The symbol commonly used to represent this is ${}_5P_3$ or ${}_nP_r$ for the number of "*n*" objects taken "*r*" at a time. Students should notice that ${}_5P_3 = 5 \times 4 \times 3$ or ${}_5P_3 = \frac{5\times4\times3\times2\times1}{2\times1}$. This can be rewritten as ${}_5P_3 = \frac{5!}{2!}$ or ${}_5P_3 = \frac{5!}{(5-3)!}$.

Generally, the number of permutations for *n* objects taken *r* at a time is given by:

$$ {}_nP_r = \frac{n(n-1)(n-2)\ldots(n-r)(n-r-1)(n-r-2)\ldots(3)(2)(1)}{(n-r)(n-r-1)(n-r-2)\ldots(3)(2)(1)} $$

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Continue to emphasize that, in the notation ${}_nP_r$, *n* is the number of elements in a set and *r* is the number of elements to be selected at any given time. Students should realize that *r* cannot be bigger than *n* since it is not possible to select more elements than are actually in the set. When using the formula ${}_nP_r = \frac{n!}{(n-r)!}$, if *r* were greater than *n*, then the denominator would contain a factorial of a negative number, which is undefined.

Students should note that the number of permutations of six people being arranged in a line is 6! This is also a permutation of a set of 6 objects from a set of 6. Therefore, applying the formula ${}_nP_r = \frac{n!}{(n-r)!}$, the result is ${}_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!}$. This means $6! = \frac{6!}{0!}$, and the only value of 0! that makes sense is 0! = 1.

Evaluating 0! means determining the number of ways there are to count an empty set. Since there is nothing to count, ask students *In how many ways can one count nothing?*. A mathematical answer to this is one.

For *n* objects taken *n* at a time, the number of permutations is $n! = {}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$. Again, 1 is the only realistic value of 0!.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  - (i) How many two-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if repetition is allowed?
    (PCBT2.3)
  - (ii) How many distinct arrangements of three letters can be formed using the letters of the word LOCKERS?
    (PCBT2.3)
  - (iii) The code for a lock consists of four numbers selected from 0, 1, 2, 3 with no repeats. For example, the code 1-2-1-3 would not be allowed but 3-0-2-1 would be allowed. Using the permutation formula, determine the number of possible codes.
    (PCBT2.4)

Journal

- Ask students to respond to the following:
  - (i) A code consists of three letters chosen from A to Z and three digits chosen from 0 to 9, with no repetitions of letters or numbers. Students should explain why the total number of possible codes can be found using the expression ${}_{26}P_3 \times {}_{10}P_3$.
    (PCBT2.3)
  - (ii) Explain why an error is obtained when trying to calculate ${}_5P_7$ on a calculator.
    (PCBT2.4)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT2 Continued ...

Achievement Indicator:

PCBT2.5 *Given a value of k, $k \in N$, solve ${}_nP_r = k$ for either n or r.*

Elaborations—Strategies for Learning and Teaching

Knowledge of permutations can be applied to solve equations of the form ${}_nP_r = k$. Students should solve equations such as the following, many of which will involve simplification of rational expressions similar to work done in Mathematics 2200 (AN5). Once the expression is simplified such that it no longer contains factorials, they are revisiting this previous work with rational expressions.

Example 1

$${}_nP_2 = 30$$

$$\frac{n!}{(n-2)!} = 30$$

$$n(n-1) = 30$$

- Students can reason that consecutive integers with a product of 30 are needed. Since $6(6 - 1) = 30$, $n = 6$.
- Alternatively, they can solve the quadratic equation $n^2 - n - 30 = 0$. This equation has roots 6 and −5. Since n must be positive, −5 is extraneous.

Example 2

$${}_{n-1}P_2 = 12$$

$$\frac{(n-1)!}{(n-1-2)!} = 12$$

$$\frac{(n-1)(n-2)(n-3)!}{(n-3)!} = 12$$

$$(n-1)(n-2) = 12$$

- Students could reason that consecutive integers with a product of 12 will satisfy this equation. Therefore, $n = 5$ because $(5 - 1)(5 - 2) = 12$.
- They could also solve the quadratic equation $n^2 - 3n + 2 = 12$ to determine that $n = 5$ (−2 is an extraneous root).

Example 3

$${}_5P_r = 5!$$

$$\frac{5!}{(5-r)!} = 5!$$

$$(5-r)! = 1$$

To solve this equation, students must recall that $1! = 1$ and $0! = 1$. Therefore, $5 - r = 1$ or $5 - r = 0$, resulting in $r = 4$, $r = 5$.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve the following:

  (i) ${}_{n}P_{2} = 42$

  (ii) ${}_{8}P_{r} = 8!$

  (iii) ${}_{n+1}P_{2} = 20$

  (iv) ${}_{n+4}P_{3} = 120$

- Ask students to answer the following:

  Denise has a set of posters to arrange on her bedroom wall. She can only fit two posters side by side. If there are 72 ways to choose and arrange two posters, how many posters does she have in total?

(PCBT2.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT2 Continued ...

Achievement Indicators:

PCBT2.6 *Explain, using examples, the effect on the total number of permutations when two or more elements are identical.*

Elaborations—Strategies for Learning and Teaching

When determining the total number of permutations with two or more identical elements, students only work with a set of *n* elements taken *n* at a time. They should be asked to solve and explain their solution to a problem such as the following:

Four beads, one blue, two red and one white, are being placed on a string. How many different arrangements are possible?

A systematic list is one strategy for determining the total number of arrangements.

Rbrw	Rrwb	wbrR	wbRr	bRrw	brRw	rbRw	rRwb
Rbwr	Rwrb	wrbR	wRbr	bRwr	brwR	rbwR	rwRb
Rrbw	Rwbr	wrRb	wRrb	bwRr	bwrR	rRbw	rwbR

Students could also recall that there are $4! = 24$ different permutations of four objects. However, there are two identical red beads, making permutations such as Rbrw and rbRw the same. If the red beads were different, they could be arranged in 2! ways. Thus, the total number of permutations with identical beads is $\frac{4!}{2!}$ or 12.

Generally, in a set of *n* objects, with *a* of one kind that are identical, *b* of a second kind that are identical, *c* of a third kind that are identical, etc., the entire set of *n* objects can be arranged in $\frac{n!}{a!b!c!\ldots}$ ways. For example, if there were one blue, two red, one white, and three black beads in the problem above, then the number of possible arrangements of all seven beads would be $\frac{7!}{2!3!} = \frac{7\times6\times5\times4\times3!}{2!3!} = 420$.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  - (i) Show that you can form 120 distinct five-letter arrangements from GREAT but only 60 distinct five-letter arrangements from GREET.
    (PCBT2.6)
  - (ii) How many distinct arrangements can be formed using all the letters of STATISTICS?
    (PCBT2.6)
  - (iii) Find the total number of arrangements of the word SILK and the total number of arrangements of the word SILL. How do your answers compare? Explain why this relationship exists.
    (PCBT2.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT2 Continued ...

Achievement Indicator:

PCBT2.7 *Solve problems involving permutations with constraints.*

Elaborations—Strategies for Learning and Teaching

A variety of constraints can be imposed in situations involving permutations. In certain situations, for example, objects must be arranged in a line where:

- two or more objects must be placed together
- two or more objects cannot be placed together
- certain object(s) must be placed in certain positions.

Students should answer questions such as the following:

How many ways can a group of five people be arranged in a line if two of them are good friends and want to sit together?

If the five people are A, B, C, D, and E, and the friends are A and B:

- The two friends can sit together as either AB or BA. Students should recognize this to be 2! ways.
- The problem can be simplified somewhat by first considering A and B as one person. Students can then temporarily think of the problem as being four different people to arrange, which can be done in 4! ways.
- By the fundamental counting principle, the five people can be arranged in $4!2! = 48$ ways.

Similarly, if three friends had wanted to sit together, the five people can be arranged in $3!3! = 36$ ways.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:

  Alice, Beatrice, Colin and Don are to be arranged in a line from left to right.

  (i) How many ways can they be arranged?

  (ii) How many ways can they be arranged if Alice and Don cannot be side by side?

  (iii) How many ways can they be arranged if Beatrice and Colin must be side by side?

  (iv) How many ways can they be arranged if Alice must be at one end of the line?

  (PCBT2.7)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.1 Permutations

SB: pp. 516-527

TR: pp. 280-286

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT3 Determine the number of combinations of *n* different elements taken *r* at a time to solve problems.

[C, PS, R, V]

Achievement Indicators:

PCBT3.1 *Explain, using examples, the differences between a permutation and a combination.*

PCBT3.2 *Determine the number of combinations of n different elements taken r at a time to solve a problem.*

Elaborations—Strategies for Learning and Teaching

In contrast to permutations, combinations are an arrangement of objects without regard for order. A formula will be developed and applied in problem solving situations.

To distinguish between permutations and combinations, students should be given a situation for each where the number of possibilities can be determined with simple counting methods.

- Adam, Marie and Brian are standing in a line at a banking machine. In how many ways could they order themselves?
- Paul, Renee and Emily are members of a committee. In how many ways could two of them be selected to attend a conference?

The essential difference between these two situations needs to be discussed and emphasized. Students should already recognize the first problem as a permutation, where order is important. Use the second problem to introduce combinations, where order is not important.

A situation such as the following could also be used to highlight the difference between permutations and combinations.

In a lottery, six numbers from 1 to 49 are selected. A winning ticket must contain the same six numbers but they may be in any order.

If order mattered, the number of permutations would be $_{49}P_6 = \frac{49!}{(49-6)!} = 10\ 068\ 347\ 520.$ Since order does not matter, the number of permutations must be divided by 6! (the number of ways of arranging the six selected numbers). The number of combinations is $_{49}C_6 = \frac{49!}{(49-6)!6!} = \frac{49\times48\times47\times46\times45\times44}{6!} = 13\ 983\ 816.$

Discuss with students why the number of combinations is less than the number of permutations.

Generally, given a set of *n* objects taken *r* at a time, the number of possible combinations is $_nC_r = \frac{n!}{(n-r)!r!}$.

Note that the notation $\binom{n}{r}$ is sometimes used instead of $_nC_r$.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Journal

- Ask students to identify which problem should be done using the fundamental counting principle, which one using the permutation formula, and which one with the combination formula. They should explain their choices.
  - (i) How many ways can a committee of three people be selected from a group of 12 people?
  - (ii) How many ways can three of eight people line up at a ticket counter?
  - (iii) How many four-digit numbers are there?

(PCBT3.1)

Paper and Pencil

- Ask students to answer the following:
  - (i) Alan, Bill, Cathy, David and Evelyn are members of the same club. Determine the number of ways of selecting:
    (a) a two-member committee
    (b) a president and a treasurer
  - (ii) Every member of the 48 students in the graduating class at a local high school would like to attend a special leadership conference, but only ten members will be allowed to attend. How many ways are there to select the lucky ten?

(PCBT3.2)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.2 Combinations

SB: pp. 528-536

TR: pp. 287-292

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT3 Continued ...

Achievement Indicators:

PCBT3.2 *Continued*

PCBT3.3 *Explain why n must be greater than or equal to r in the notation* ${}_nC_r$ *or* $\binom{n}{r}$.

PCBT3.4 *Explain, using examples, why* ${}_nC_r = {}_nC_{n-r}$ *or* $\binom{n}{r} = \binom{n}{n-r}$.

PCBT3.5 *Given a value of k, k ϵ N, solve* ${}_nC_r = k$ *or* $\binom{n}{r} = k$ *for either n or r.*

Elaborations—Strategies for Learning and Teaching

Students should also solve problems such as the following, where they apply both combinations and the fundamental counting principle.

A baseball team has 5 pitchers, 6 outfielders and 10 infielders. For a game, the manager needs to field a starting group with 1 pitcher, 3 outfielders and 5 infielders. How many ways can she select the starting group?

- There are ${}_5C_1$, or 5, ways to select a pitcher.
- There are ${}_6C_3$, or 20, ways to select the outfielders.
- There are ${}_{10}C_5$, or 252 ways to select the infielders.

Students can apply the fundamental counting principle to determine that there are 25 200 ways to select the starting group.

Students should be able to easily verify that the value of *n* must be greater than or equal to *r* in ${}_nC_r$, having explained the reasoning when working with permutations. If they understand that the notation ${}_nC_r$ means choosing *r* elements from a set of *n* elements, it naturally flows that *r* cannot be larger than *n*.

To explore why ${}_nC_r = {}_nC_{n-r}$, a problem such as the following could be used:

Suppose the letters A, B, C, D and E are placed in a bag and two letters are selected at random. How many combinations of letters can be selected?

Most students will determine the number of combinations of two letters to be ${}_5C_2 = \frac{5!}{3!2!}$. Prompt them to think of a different way to solve the problem. It can be thought of in “reverse”. Each time two letters are selected, three letters are also being selected to remain in the bag. The number of combinations of three letters is ${}_5C_3 = \frac{5!}{2!3!}$. Students should conclude from this that ${}_5C_2 = {}_5C_{5-2}$ and generally ${}_nC_r = {}_nC_{n-r}$. An algebraic proof of this result can be presented but is not required knowledge for students.

Students are expected to solve a variety of equations for either *n* or *r*.

$${}_nC_2 = 15 \qquad {}_6C_r = 15 \qquad {}_nC_{n-2} = 15 \qquad \binom{n}{2} = 15$$

To solve an equation such as ${}_6C_r = 15$, students should realize that $r \leq 6$ since *r* elements are being chosen from a set of 6 elements. Checking each of the possibilities results in the solution $r = 2$, $r = 4$. For this type of equation, manageable numbers that make trial-and-error a reasonable strategy should be used.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  (i) A set of flash cards consists of 13 red, 13 blue, 13 black and 13 yellow cards. The cards in each colour are numbered from 1 through 13.
    (a) How many groups of 5 cards can be selected from the entire set?
    (b) How many groups of 5 cards can be selected from the red cards?
    (c) How many groups of 20 cards can be selected from the entire set if there must be five of each colour?

  (ii) How many different sums of money can be made by selecting four coins out of a set consisting of a nickel, a dime, a quarter, a dollar coin, and a two-dollar coin?

(PCBT3.2)

- Ask students to solve the following:
  (i) $\binom{n+1}{1} = 20$
  (ii) ${}_7C_r = 21$
  (iv) ${}_{n+1}C_{n-1} = 6$

(PCBT3.5)

Journal

- Ask students to explain why an error is obtained when attempting to calculate ${}_5C_7$ on a calculator.

(PCBT3.3)

- Ask students to explain why the number of ways of selecting eight people from a group of ten is equal to the number of ways of selecting two people from a group of 10.

  Then they should discuss whether the number of ways of selecting eight or two people would be equal if this was a permutation problem where the selected people had to be lined up in certain positions.

(PCBT3.4)

- Ask students to explain how they could solve the equation $6({}_nC_3) = 6$ using their understanding of combinations rather than an algebraic solution.

(PCBT3.5)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.2 Combinations

SB: pp. 528-536

TR: pp. 287-292

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT4 Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

[CN, R, V]

Achievement Indicator:

PCBT4.1 *Explain the patterns found in the expanded form of $(x + y)^n$, $n \leq 4$ and $n \in N$, by multiplying n factors of $(x + y)$.*

PCBT4.2 *Explain how to determine the subsequent row in Pascal's triangle, given any row.*

Elaborations—Strategies for Learning and Teaching

Students expand a binomial using Pascal's triangle and the binomial theorem. They see that if Pascal's Triangle has already been completed for the desired expansion, the appropriate row can be read with no computation required. They should also see that a disadvantage in using Pascal's triangle is that each of the preceding rows in the display must be completed before the row needed for an expansion can be obtained. This leads to the development of the binomial theorem.

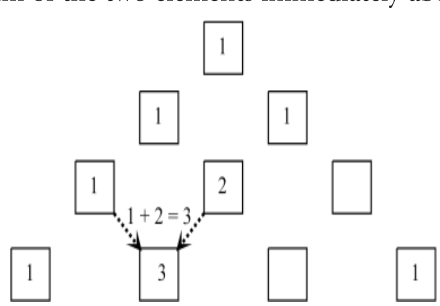
In Mathematics 1201, students multiplied polynomial expressions, including monomials, binomials and trinomials (AN4). They should perform the following binomial expansions algebraically and record the coefficients of the terms:

Expression	Coefficients of the Terms
$(x + y)^0 = 1$	
$(x + y)^1 = x + y$	
$(x + y)^2 = x^2 + 2xy + y^2$	
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$	
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$	

Students should look for patterns in the exponents of the terms for each expansion.

- How is the exponent of the binomial related to the exponents of the first and last terms in the expansions?
- Moving from left to right in each expansion, what patterns are there in the exponents for x and y?

They should also look for patterns in the triangle. For example, each element is the sum of the two elements immediately above it.



This activity develops the elements in rows 1 through 5 for Pascal's triangle, a triangular array of numbers described by Blaise Pascal in 1653. From the activity, students should be able to explain how to determine the elements in any row once the preceding row is known.

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Journal

- Ask students to describe the pattern that the exponents of x and y follow when doing an expansion such as $(x + y)^4$.

(PCBT4.1)

- Ask students to respond to the following:

  Suppose a friend has missed the class in which you learned to construct Pascal's triangle. Write a point-form description of the steps you would tell her to follow in order to construct the triangle on her own.

(PCBT4.2)

Paper and Pencil

- Based on what was observed for expansions with powers up to $n = 4$, ask students to predict the exponents for the 5^{th} term in the expansion of $(x + y)^7$.

(PCBT4.1)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.3 The Binomial Theorem

SB: pp. 537-545

TR: pp. 293-297

Permutations, Combinations and The Binomial Theorem

Outcomes

Students will be expected to

PCBT4 Continued ...

Achievement Indicators:

PCBT4.3 *Relate the coefficients of the terms in the expansion of $(x + y)^n$ to the $(n + 1)$ row in Pascal's triangle.*

PCBT4.4 *Explain, using examples, how the coefficients of the terms in the expansion of $(x + y)^n$ are determined by combinations.*

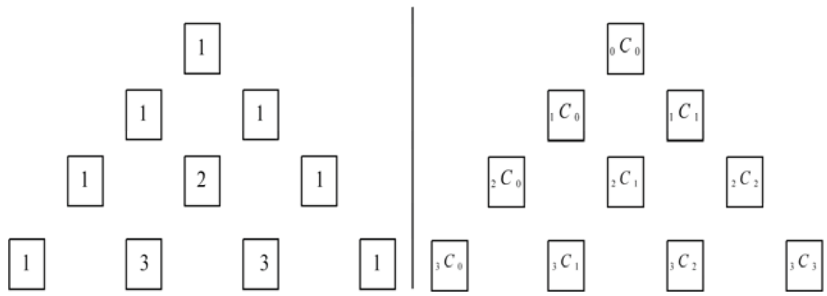
PCBT4.5 *Expand, using the binomial theorem, $(x + y)^n$.*

PCBT4.6 *Determine a specific term in a binomial expansion.*

Elaborations—Strategies for Learning and Teaching

Using Pascal's triangle, students should now also be able to find the expansion for $(x + y)^n$ for any value of n where $n \leq 12$.

The elements in each row of Pascal's triangle can be determined using the formula for ${}_nC_r$, as shown in the diagram.



Using combinations, students should be able to predict the element or the coefficient of any element in the expansion of $(x + y)^n$. In the expansion of $(x + y)^7$, for example, the coefficient of x^3y^4 is ${}_7C_4 = 35$. Generally, the expansion of $(x + y)^n$ is:

$$ {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n $$

This is known as the binomial theorem.

Students should now apply the binomial theorem to expansions of other binomial expressions, such as:

- $(2a + b)^7$
- $\left(\frac{x}{3} - 2\right)^5$
- $\left(x + \frac{1}{x^3}\right)^{10}$

General Outcome: Develop algebraic and numeric reasoning that involves combinatorics.

Suggested Assessment Strategies

Journal

- Ask students to respond to the following:

  (i) Explain how using Pascal's triangle makes expanding binomials easier when using larger exponents.

  (ii) Would there be any disadvantages to using Pascal's triangle if you wanted to find a single term in a binomial expansion with a very large exponent?

(PCBT4.3)

Paper and Pencil

- Ask students to answer the following:

  (i) The combination formula is used to find a particular coefficient in a binomial expansion.
  (a) Which row and entry in Pascal's triangle could be used to accomplish the same task?
  (b) What is the exponent being used to expand the binomial?
  (c) For which term in the expansion will this give the coefficient?
  (d) What will be the exponents of the variables for this particular term?
  (e) What will be the largest coefficient in this particular expansion?

(PCBT 4.4)

  (ii) Expand and simplify each of the following:
  (a) $(a+3b)^4$
  (b) $(1-5y)^3$
  (c) $\left(\frac{x}{2}+4\right)^6$
  (d) $\left(x^2-\frac{3}{x^2}\right)^{12}$

(PCBT 4.5)

  (iii) Determine the middle term in the expansion of $(2c\ \ 3d)^6$.

(PCBT 4.6)

  (iv) Find the indicated term in each of the following expansions:
  (a) The second term of $(5 + x)^6$
  (b) The fifth term of $(x + 7)^7$
  (c) The third term of $(x - 2)^6$
  (d) The third term of $(2x + 3y)^7$
  (e) The fourth term of $(3x - 7y)^5$

(PCBT 4.6)

Resources/Notes

Authorized Resource

Pre-Calculus 12

11.3 The Binomial Theorem

SB: pp. 537-545

TR: pp. 293-297

Appendix:

Outcomes with Achievement Indicators
Organized by Topic
(With Curriculum Guide Page References)

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF1. Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V]	RF1.1 Compare the graphs of a set of functions of the form $y - k = f(x)$ to the graph of $y = f(x)$ and generalize, using inductive reasoning, a rule about the effect of k.	p. 42
	RF1.2 Compare the graphs of a set of functions of the form $y = f(x - \text{h})$ to the graph of $y = f(x)$ and generalize, using inductive reasoning, a rule about the effect of h.	p. 42
	RF1.3 Compare the graphs of a set of functions of the form $y - k = f(x - \text{h})$ to the graph of $y = f(x)$ and generalize, using inductive reasoning, a rule about the effect of h and k.	p. 42
	RF1.4 Sketch the graph of $y - k = f(x)$, $y = f(x - \text{h})$ or $y - k = f(x - \text{h})$ for given values of h and k, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 42
	RF1.5 Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y = f(x)$.	p. 44
RF2. Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V]	RF2.1 Compare the graphs of a set of functions of the form $y = af(x)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of a.	p. 48
	RF2.2 Compare the graphs of a set of functions of the form $y = f(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of b.	p. 50
	RF2.3 Compare the graphs of a set of functions of the form $y = af(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effects of a and b.	p. 52
	RF2.4 Sketch the graph of $y = af(x)$, $y = f(bx)$ or $y = af(bx)$ for given values of a and b, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 54
	RF2.5 Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.	p. 54

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF3. Apply translations and stretches to the graphs and equations of functions. [C, CN, R, V]	RF3.1 Sketch the graph of the function $y - k = af(b(x - \text{h}))$ for given values of a, b, h and k, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 56
	RF3.2 Write the equation of a function, given its graph which is a translation and/or stretch of the graph of the function $y = f(x)$.	p. 58
RF4. Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the: • x-axis • y-axis • line $y = x$. [C, CN, R, V]	RF4.1 Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection through the x-axis or the y-axis.	p. 46
	RF4.2 Sketch the reflection of the graph of a function $y = f(x)$ through the x-axis or the y-axis, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 46
	RF4.3 Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ through the x-axis or the y-axis.	p. 46
	RF4.4 Sketch the graphs of the functions $y = -f(x)$ and $y = f(-x)$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 46
	RF4.5 Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$ through the x-axis or the y-axis.	p. 46
	RF4.6 Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection through the line $y = x$.	p. 60
	RF4.7 Sketch the reflection of the graph of a function $y = f(x)$ through the line $y = x$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.	p. 62
	RF4.8 Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ through the line $y = x$.	p. 62
	RF4.9 Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$ through the line $y = x$.	p. 66

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF5. Demonstrate an understanding of inverses of relations. [C, CN, R, V]	RF5.1 Explain how the transformation $(x, y) \rightarrow (y, x)$ can be used to sketch the inverse of a relation.	p. 60
	RF5.2 Explain the relationship between the domains and ranges of a relation and its inverse.	p. 60
	RF5.3 Explain how the graph of the line $y = x$ can be used to sketch the inverse of a relation.	p. 62
	RF5.4 Sketch the graph of the inverse relation, given the graph of a relation.	p. 62
	RF5.5 Determine if a relation and its inverse are functions.	p. 64
	RF5.6 Determine restrictions on the domain of a function in order for its inverse to be a function.	p. 66
	RF5.7 Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.	p. 66
	RF5.8 Determine, algebraically or graphically, if two functions are inverses of each other.	p. 66
RF6. Demonstrate an understanding of logarithms. [CN, ME, R]	RF6.1 Explain the relationship between logarithms and exponents.	p. 160
	RF6.2 Express a logarithmic equation as an exponential equation and vice versa.	p. 160
	RF6.3 Determine, without technology, the exact value of a logarithm, such as $\log_2 8$.	p. 160
	RF6.4 Estimate the value of a logarithm, using benchmarks, and explain the reasoning; e.g., since $\log_2 8 = 3$ and $\log_2 16 = 4$, $\log_2 9$ is approximately equal to 3.1.	p. 160
RF7. Demonstrate an understanding of the product, quotient and power laws of logarithms. [C, CN, ME, R, T]	RF7.1 Develop and generalize the laws of logarithms, using numeric examples and exponent laws.	p. 168
	RF7.2 Derive each law of logarithms.	p. 168
	RF7.3 Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.	p. 170
	RF7.4 Determine, with technology, the approximate value of a logarithmic expression, such as $\log_2 9$.	p. 170

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF8. Graph and analyze exponential and logarithmic functions. [C, CN, T, V]	RF8.1 Sketch, with or without technology, a graph of an exponential function of the form $y = c^x$, $c > 0$, $c \neq 1$.	p. 142-144
	RF8.2 Identify the characteristics of the graph of an exponential function of the form $y = c^x$, $c > 0$, $c \neq 1$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.	p. 142-144
	RF8.3 Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = c^x$, $c > 0$, $c \neq 1$ and state the characteristics of the graph.	p. 146
	RF8.4 Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.	p. 162
	RF8.5 Sketch with or without technology, the graph of a logarithmic function of the form $y = \log_c x$, $c > 1$.	p. 164
	RF8.6 Identify the characteristics of the graph of a logarithmic function of the form $y = \log_c x$, $c > 1$, including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.	pp. 164-166
	RF8.7 Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, $c > 1$, and state the characteristics of the graph.	pp. 164-166

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF9. Solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]	RF9.1 Determine the solution of an exponential equation for which both sides can be written as rational powers of the same base.	p. 148
	RF9.2 Determine the solution of an exponential equation in which the bases are not rational powers of one another, using a variety of strategies.	p. 148-150, 172
	RF9.3 Solve a problem that involves exponential growth or decay.	p. 152-154, 174
	RF9.4 Solve a problem that involves the application of exponential equations to loans, mortgages and investments.	p. 152-154, 176
	RF9.5 Solve a problem by modeling a situation with an exponential or a logarithmic equation.	p. 152-154, 174-176
	RF9.6 Determine the solution of a logarithmic equation, and verify the solution.	p. 172
	RF9.7 Explain why a value obtained in solving a logarithmic equation may be extraneous.	p. 172
	RF9.8 Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.	p. 174
RF10. Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME]	RF10.1 Explain how long division of a polynomial expression by a binomial expression of the form $x - a$, $a \in I$, is related to synthetic division.	p. 26
	RF10.2 Divide a polynomial expression by a binomial expression of the form $x - a$, $a \in I$ using long division or synthetic division.	pp. 26-28
	RF10.3 Explain the relationship between the remainder when a polynomial expression is divided by $x - a$, $a \in I$, and the value of the polynomial expression at $x = a$ (remainder theorem).	p. 28
	RF10.4 Explain and apply the factor theorem to express a polynomial expression as a product of factors.	pp. 28-30
	RF10.5 Explain the relationship between linear factors of a polynomial expression and the zeros of the corresponding polynomial function.	pp. 28-30

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations..	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF11. Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]	RF11.1 Identify the polynomial functions in a set of functions, and explain the reasoning.	p. 22
	RF11.2 Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.	p. 22
	RF11.3 Generalize rules for graphing polynomial functions of odd or even degree.	p. 24
	RF11.4 Explain the relationship among the following: • the zeros of a polynomial function • the roots of the corresponding polynomial equation • the *x*-intercepts of the graph of the polynomial function.	p. 30
	RF11.5 Explain how the multiplicity of a zero of a polynomial function affects the graph.	p. 32
	RF11.6 Sketch, with or without technology, the graph of a polynomial function.	p. 34
	RF11.7 Solve a problem by modeling a given situation with a polynomial function.	p. 36
	RF11.8 Determine the equation of a polynomial function given its graph.	p. 36
RF12. Graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]	RF12.1 Sketch the graph of the function $y=\sqrt{x}$, using a table of values, and state the domain and range.	p. 72
	RF12.2 Sketch the graph of the function $y-k=a\sqrt{b(x-h)}$ by applying transformations to the graph of the function $y=\sqrt{x}$, and state the domain and range.	pp. 72-74
	RF12.3 Sketch the graph of the function $y=\sqrt{f(x)}$, given the equation or graph of the function $y = f(x)$, and explain the strategies used.	pp. 76-78
	RF12.4 Compare the domain and range of the function $y=\sqrt{f(x)}$ to the domain and range of the function $y = f(x)$, and explain why the domains and ranges may differ.	pp. 76-78
	RF12.5 Describe the relationship between the roots of a radical equation and the *x*-intercepts of the graph of the corresponding radical function.	p. 78
	RF12.6 Determine, graphically, an approximate solution of a radical equation.	p. 78

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T1. Demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]	T1.1 Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.	p. 84
	T1.2 Sketch, in standard position, an angle with a measure of 1 radian.	p. 84
	T1.3 Describe the relationship between radian measure and degree measure.	pp. 84-86
	T1.4 Sketch, in standard position, an angle with a measure expressed in the form $k\pi$ radians, where $k \in Q$.	p. 86
	T1.5 Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.	p. 86
	T1.6 Express the measure of an angle in degrees, (exact value or decimal approximation) given its measure in radians.	p. 86
	T1.7 Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.	p. 88
	T1.8 Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.	p. 88
	T1.9 Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius r, and solve problems based upon that relationship.	p. 88
T2. Develop and apply the equation of the unit circle. [CN, R, V]	T2.1 Derive the equation of the unit circle from the Pythagorean theorem.	p. 90
	T2.2 Generalize the equation of a circle with centre (0, 0) and radius r.	p. 90
	T2.3 Describe the six trigonometric ratios, using a point $P(x, y)$ that is the intersection of the terminal arm of an angle and the unit circle.	p. 90

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T3. Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]	T3.1 Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either degrees or radians.	p. 92
	T3.2 Determine, using a unit circle or reference triangle, the exact value of a trigonometric ratio for angles expressed in degrees that are multiples of 0˚, 30˚, 45˚, 60˚ or 90˚, or for angles expressed in radians that multiples of 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$ and explain the strategy.	p. 92
	T3.3 Sketch a diagram to represent a problem that involves trigonometric ratios.	p. 92
	T3.4 Determine, with or without technology, the measures, in degrees or radians, of the angles in a specified domain, given the value of a trigonoemtric ratio.	p. 94
	T3.5 Describe the six trigonometric ratios, using a point P(x, y) that is the intersection of the terminal arm of an angle and the unit circle.	p. 94
	T3.6 Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.	p. 94
	T3.7 Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.	p. 96
	T3.8 Solve a problem, using trigonometric ratios.	p. 96

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T4. Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS, T, V]	T4.1 Sketch, with or without technology, the graph of $y = \sin x$ and $y = \cos x$.	p. 106
	T4.2 Determine the characteristics (amplitude, domain, period, range and zeros) of the graph of $y = \sin x$ and $y = \cos x$.	p. 106
	T4.3 Determine how varying the value of a affects the graph of $y = a \sin x$ and $y = a \cos x$.	p. 108
	T4.4 Determine how varying the value of b affects the graph of $y = \sin bx$ and $y = \cos bx$.	p. 108
	T4.5 Determine how varying the value of d affects the graph of $y = \sin x + d$ and $y = \cos x + d$.	p. 108
	T4.6 Determine how varying the value of c affects the graph of $y = \sin(x + c)$ and $y = \cos(x + c)$.	p. 108
	T4.7 Sketch, without technology, graphs of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$ using transformations, and explain the strategies.	p. 108
	T4.8 Determine the characteristics (amplitude, domain, period, phase shift, range and zeros) of the graph of a trigonometric function of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$.	p. 108
	T4.9 Determine the values of a, b, c and d for functions of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$ that correspond to a given graph, and write the equation of the function.	p. 110
	T4.10 Solve a given problem by analyzing the graph of a trigonometric function.	p. 110
	T4.11 Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.	p. 110
	T4.12 Determine a trigonometric function that models a situation to solve a problem.	p. 110
	T4.13 Sketch, with or without technology, the graph of $y = \tan x$.	p. 112
	T4.14 Determine the characteristics (asymptotes, domain, period, range and zeros) of the graph of $y = \tan x$.	p. 112

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T5. Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]	T5.1 Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible. T5.2 Determine, using technology, the approximate solution of a trigonometric equation. T5.3 Verify, with or without technology, that a given value is a solution to a trigonometric equation. T5.4 Identify and correct errors in a solution for a trigonometric equation. T5.5 Relate the general solution of a trigonometric equation to the zeros of the corresponding function (restricted to sine and cosine functions).	pp. 98, 114, 134 pp. 98, 114, 136 pp. 98, 114, 136 pp. 100, 136 p. 114
T6. Prove trigonometric identities, using: • reciprocal identities • quotient identities • Pythagorean identities • sum or difference identities (restricted to sine, cosine and tangent) • double-angle identities (restricted to sine, cosine and tangent). [R, T, V]	T6.1 Explain the difference between a trigonometric identity and a trigonometric equation. T6.2 Determine, graphically, the potential validity of a trigonometric identity, using technology. T6.3 Determine the non-permissible values of a trigonometric identity. T6.4 Verify a trigonometric identity numerically for a given value in either degrees or radians. T6.5 Prove, algebraically, that a trigonometric identity is valid. T6.6 Simplify trigonometric expressions using trigonometric identities. T6.7 Determine, using the sum, difference and double-angle identities, the exact value of a trigonometric ratio. T6.8 Explain why verifying that the two sides of a trigonometric identity are equal for given values is insufficient to conclude that the identity is valid.	pp. 120-122 pp. 122, 128-130 pp. 122, 128-132 pp. 124, 128-130 pp. 124, 130-132 pp. 126-130 pp. 128-130 p. 132

Topic: Permutations, Combinations and the Binomial Theorem	**General Outcome**: Develop algebraic and numeric reasoning that involves combinatorics.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
PCBT1. Apply the fundamental counting principle to solve problems. [C, PS, R, V]	PCBT1.1 Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.	pp. 182-184
	PCBT1.2 Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.	p.186
	PCBT1.3 Solve a simple counting problem by applying the fundamental counting principle.	p. 186
PCBT2. Determine the number of permutations of n elements taken r at a time to solve problems. [C, PS, R, V]	PCBT2.1 Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.	p. 188
	PCBT2.2 Determine, in factorial notation, the number of permutations of n different elements taken n at a time to solve a problem.	p. 188
	PCBT2.3 Determine, using a variety of strategies, the number of permutations of n different elements taken r at a time to solve a problem.	p. 190
	PCBT2.4 Explain why n must be greater than or equal to r in the notation ${}_nP_r$.	p. 190
	PCBT2.5 Given a value of k, $k \in \mathrm{N}$, solve ${}_nP_r = k$ for either n or r.	p. 192
	PCBT2.6 Explain, using examples, the effect on the total number of permutations when two or more elements are identical.	p. 194
	PCBT2.7 Solve problems involving permutations with constraints.	p. 196

Topic: Permutations, Combinations and the Binomial Theorem	**General Outcome**: Develop algebraic and numeric reasoning that involves combinatorics.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
PCBT3. Determine the number of combinations of *n* different elements taken *r* at a time to solve problems. [C, PS, R, V]	PCBT3.1 Explain, using examples, the differences between a permutation and a combination.	p. 198
	PCBT3.2 Determine the number of combinations of *n* different elements taken *r* at a time to solve a problem.	pp. 198-200
	PCBT3.3 Explain why *n* must be greater than or equal to *r* in the notation ${}_nC_r$ or $\binom{n}{r}$.	p. 200
	PCBT3.4 Explain, using examples, why ${}_nC_r = {}_nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$.	p. 200
	PCBT3.5 Given a value of k, $k \in \mathrm{N}$, solve ${}_nC_r = k$ or $\binom{n}{r} = k$ for either *n* or *r*.	p. 200
PCBT4. Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]	PCBT4.1 Explain the patterns found in the expanded form of $(x + y)^n$, $n \leq 4$ and $n \in \mathrm{N}$, by multiplying *n* factors of $(x + y)$.	p. 202
	PCBT4.2 Explain how to determine the subsequent row in Pascal's triangle, given any row.	p. 202
	PCBT4.3 Relate the coefficients of the terms in the expansion of $(x + y)^n$ to the $(n + 1)$ row in Pascal's triangle.	p. 204
	PCBT4.4 Explain, using examples, how the coefficients of the terms in the expansion of $(x + y)^n$ are determined by combinations.	p. 204
	PCBT4.5 Expand, using the binomial theorem, $(x + y)^n$.	p. 204
	PCBT4.6 Determine a specific term in a binomial expansion.	p. 204

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